

Effective de Rham Cohomology – The General Case

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Abstract

Grothendieck has proved that each class in the de Rham cohomology of a smooth complex affine variety can be represented by a differential form with polynomial coefficients. After having proved a single exponential bound for the degrees of these forms in the case of a hypersurface, here we generalize this result to arbitrary codimension. More precisely, we show that the p -th de Rham cohomology of a smooth affine variety of dimension m and degree D can be represented by differential forms of degree $(pD)^{O(pm)}$. This result is relevant for the algorithmic computation of the cohomology, but is also motivated by questions in the theory of ordinary differential equations related to the infinitesimal Hilbert 16th problem.

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1 Introduction

Let X be a smooth variety in \mathbb{C}^n . A fundamental result of Grothendieck says that the cohomology of X can be described in terms of *algebraic* differential forms on X [18]. More precisely, he proved that the singular cohomology of X is isomorphic to the algebraic de Rham cohomology $H_{\mathrm{dR}}^\bullet(X)$, which is defined as the cohomology of the complex of algebraic differential forms on X . Hence, each cohomology class in $H_{\mathrm{dR}}^p(X)$ can be represented by a p -form

$$\omega = \sum_{i_1 < \dots < i_p} \omega_{i_1 \dots i_p} dX_{i_1} \wedge \dots \wedge dX_{i_p}, \quad (1)$$

where the $\omega_{i_1 \dots i_p}$ are polynomial functions on X . However, Grothendieck's proof is not *effective*, i.e., it gives no information about the degrees of the polynomials $\omega_{i_1 \dots i_p}$. In [36] we proved a single exponential bound on their degrees in the case that X is a hypersurface. In particular, if $X \subseteq \mathbb{C}^n$ is a

smooth hypersurface of degree D , then each cohomology class in $H_{\text{dR}}^p(X)$ can be represented by a differential form ω as in (1), where the $\omega_{i_1 \dots i_p}$ are polynomials of degree $D^{\mathcal{O}(pn)}$. The present paper generalizes this result to arbitrary smooth varieties.

1.1 Motivation

It is a long standing open question in algorithmic real algebraic geometry to find a single exponential time algorithm for computing the Betti numbers of a semialgebraic set. Single exponential time algorithms are known, e.g., for counting the connected components and computing the Euler characteristic of a semialgebraic set (for an overview see [2], for details and exhaustive bibliography see [3]). The best result in this direction states that for fixed ℓ one can compute the first ℓ Betti numbers of a semialgebraic set in single exponential time [1].

Over the complex numbers, one approach for computing Betti numbers is to compute the algebraic de Rham cohomology. In [30, 37] the de Rham cohomology of the complement of a complex affine variety is computed using Gröbner bases for \mathcal{D} -modules. This algorithm is extended in [38] to compute the cohomology of a projective variety. However, the complexity of these algorithms is not analyzed, and due to their use of Gröbner bases a good worst-case complexity is not to be expected. In [7] a single exponential time (in fact, parallel polynomial time) algorithm is given for counting the connected components, i.e., computing the zeroth de Rham cohomology, of a (possibly singular) complex variety. This algorithm is extended in [35] to one with the same asymptotic complexity for computing equations for the components. The first single exponential time algorithm for computing *all* Betti numbers of an interesting class of varieties is given in [34]. Namely, this paper shows how to compute the de Rham cohomology of a smooth projective variety in parallel polynomial time. In terms of structural complexity in the Turing model, these results are the best one can hope for, since the problem of computing a fixed Betti number (e.g., deciding connectedness) of a complex affine or projective variety defined over the integers is PSPACE-hard [33].

Besides being relevant for algorithms, our question also has connections to the theory of ordinary differential equations. The long standing infinitesimal Hilbert 16th problem has been solved in [5]. The authors derive a bound on the number of limit cycles generated from nonsingular energy level ovals (isolated periodic trajectories) in a non-conservative perturbation of a Hamiltonian polynomial vector field in the plane. It seems that their proof can be considerably generalized to solutions of certain linear systems of Pfaffian differential equations. Examples of such systems are provided by period matrices of polynomial maps, once the corresponding Gauss-Manin connexion can be explicitly constructed. For this construction one needs degree bounds for generators of the cohomology of the generic fibers of the polynomial map.

1.2 Known Cases

We have shown in [7, Theorem 3.3] that the zeroth de Rham cohomology of X is isomorphic to its zeroth singular cohomology even when X is singular, and that this cohomology has a basis of degree $d^{\mathcal{O}(n^2)}$, if X is defined by polynomials of degree $\leq d$.

It follows from the results of [34] that if X has no singularities at infinity, i.e., the projective closure of X in \mathbb{P}^n is smooth, then each class in $H_{\text{dR}}^\bullet(X)$ can be represented by a differential form of degree at most $m(em + 1)D$, where $m = \dim X$, $D = \deg X$, and e is the maximal codimension of the irreducible components of X . However, in general X does have singularities at infinity, and resolution of singularities has a very bad worst-case complexity [4].

Another special case with known degree bounds is the complement of a projective hypersurface, which we will actually use in this paper (see the proof of Theorem 7.1). The statement follows essentially from [9] and [10], the argument can be found in [12, Corollary 6.1.32], see also [11]. Let $f \in \mathbb{C}[X_0, \dots, X_n]$ be a homogeneous polynomial, and consider $U := \mathbb{P}^n \setminus \mathcal{Z}(f)$, which is an affine variety. Then, each class in $H_{\text{dR}}^p(U)$ is represented by a (homogeneous) differential form

$$\frac{\alpha}{f^p} \quad \text{with} \quad \deg \alpha = p \deg f$$

(see §2.1 for the definition of the degree of a differential form). Since this result was already proved by Griffiths in a special case [16], we call it the Griffiths-Deligne-Dimca (GDD) bound.

1.3 Main Result

The main result of this paper is that the algebraic de Rham cohomology of a smooth affine variety can be represented by differential forms of single exponential degree. More precisely, we prove the following.

Theorem 1.1. *Let k be an algebraically closed field of characteristic zero, and let X be a smooth affine m -dimensional variety of degree D over k . Then each cohomology class in $H_{\text{dR}}^p(X)$ can be represented by a differential p -form of degree at most*

$$2^{2pm+6m+2} p^{2pm+6m+1} D^{4pm+10m+1} + D^{m+1} = (pD)^{\mathcal{O}(pm)}.$$

Remark 1.2. (i) Note that we kept the additive term D^{m+1} , so that the bound is valid for $p = 0$ as well (cf. Proposition 3.1).

(ii) If $X \subseteq \mathbb{A}^n$, then the term D^{m+1} can be replaced by $\frac{n+1}{4} D^2$ (Corollary 4.1).

(iii) For a hypersurface, the slightly better bound of $D^{\mathcal{O}(pm)}$ is proved in [36].

1.4 Proof Ideas

The proof of Theorem 1.1 consists of two major steps. In a first step we reduce the question to the case of an irreducible locally closed hypersurface, and in a second step we prove a bound for this special case (Theorem 7.1).

The reduction to irreducible X (Corollary 4.1) is an easy consequence of our characterization of the zeroth de Rham cohomology and the bound from [7]. In the present paper, using newer versions of the effective Nullstellensatz due to Kollár [26] and Jelonek [24], we improve this bound in two directions (Propositions 3.1 and 3.2). We also give an example showing that the second version is sharp up to a factor of $n + 1$, where n is the ambient dimension.

Then, if X is irreducible of dimension m , a generic projection to a linear subspace of dimension $m + 1$ restricts to a birational map from X to a hypersurface Y . By Zariski's Main Theorem, there exist open dense subsets $U \subseteq X$ and $V \subseteq Y$, such that the projection maps U isomorphically onto V . Using a *geometric resolution* of X [14, 31], we construct these locally closed sets effectively. More precisely, there exists a polynomial f of degree $\leq D$ in $m + 1$ variables, such that $U = X \setminus \mathcal{Z}(g)$ and $V = \mathcal{Z}(f) \setminus \mathcal{Z}(g)$, where g is a partial derivative of f (Lemma 4.2). It follows that a degree bound for $H_{\mathrm{dR}}^\bullet(V)$ implies a degree bound for $H_{\mathrm{dR}}^\bullet(U)$ (Corollary 4.4). Moreover, it is also not difficult to see that one can cover X by such principal open subsets U_i (Corollary 4.5). To finish the first major step, it remains to show how degree bounds for the open patches yield a bound for $H_{\mathrm{dR}}^\bullet(X)$ (Theorem 5.3). This is done using the concept of hypercohomology, which through a spectral sequence immediately yields an effective description of $H_{\mathrm{dR}}^\bullet(X)$ (Lemma 5.2) in terms of the total complex of the Čech-de Rham double complex. In order to derive a bound for the usual description in terms of global sections, we make the Čech cohomology effective (Lemma 5.1). In this construction we make essential use of Jelonek's effective Nullstellensatz [24].

The main idea for the treatment of the locally closed hypersurface V is the same as in [36], namely to prove an effective version of the Gysin sequence (Theorem 6.1) and use the GDD bound for complements of hypersurfaces mentioned in §1.2. However, in [36] we considered the case of a closed hypersurface $X \subseteq \mathbb{A}^n$, in which case the Gysin sequence yields an isomorphism $H_{\mathrm{dR}}^p(\mathbb{A}^n \setminus X) \xrightarrow{\sim} H_{\mathrm{dR}}^{p-1}(X)$, which we made effective. Here, we have only a locally closed subset $V = \mathcal{Z}(f) \setminus \mathcal{Z}(g)$ of \mathbb{A}^n . One idea is to treat V as a closed subset of $\mathbb{A}^n \setminus \mathcal{Z}(g)$. Though we are able to construct an effective residue map in this setting, this map may not be surjective, since the surrounding space $\mathbb{A}^n \setminus \mathcal{Z}(g)$ has non-trivial cohomology in general. Our solution is to consider V as a codimension 2 complete intersection $Z \subseteq \mathbb{A}^{n+1}$ through the isomorphism $\mathbb{A}^n \setminus \mathcal{Z}(g) \simeq \mathcal{Z}(gY - 1) \subseteq \mathbb{A}^{n+1}$. Since the Gysin sequence in the version of [19] does not hinge on codimension 1, we get an isomorphism $H_{\mathrm{dR}}^{p+1}(W) \xrightarrow{\sim} H_{\mathrm{dR}}^{p-2}(Z)$, where $W := \mathbb{A}^{n+1} \setminus Z$ (Corollary 6.2). Luckily, while the general complete intersection case seems considerably more difficult, we are able to prove the crucial Lemma (Lemma 6.4) exactly in this special case and obtain an effective Gysin sequence (Theorem 6.3).

However, the price to pay is that now the complement W is not affine anymore, so we need hypercohomology also to realize the de Rham cohomology of W , and a spectral sequence argument (Lemma 5.2) to apply the GDD bound (Lemma 5.4). Using local cohomology and another spectral sequence argument, we get a more succinct description of $H_{\text{dR}}^\bullet(W)$ in terms of sheaf cohomology (Lemma 5.5).

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2 Preliminaries

2.1 Basic Notations and Facts

Throughout this paper, let k be an algebraically closed field of characteristic zero, and $R := k[X_1, \dots, X_n]$. An *algebraic set* or *closed variety* is the common zero set of polynomials $f_1, \dots, f_r \in R$ in the affine space $\mathbb{A}^n = k^n$, i.e.,

$$X = \mathcal{Z}(f_1, \dots, f_r) = \{x \in k^n \mid f_1(x) = \dots = f_r(x) = 0\}.$$

Note that X may be reducible. More generally, the term *variety* will refer to a *locally closed set*, i.e., a Zariski open subset $X \subseteq Y$ of a closed variety $Y \subseteq \mathbb{A}^n$. A variety X is called *affine* iff it is a *principal open subset* of Y , i.e.,

$$X = Y \setminus \mathcal{Z}(g) = \mathcal{Z}(f_1, \dots, f_r) \setminus \mathcal{Z}(g), \quad f_1, \dots, f_r, g \in R.$$

Indeed, these are exactly the varieties which are isomorphic to a closed variety, namely $\mathcal{Z}(f_1, \dots, f_r, X_{n+1}g - 1) \subseteq \mathbb{A}^{n+1}$. The (vanishing) ideal of an affine variety X is defined as $I(X) = \{f \in R \mid f(x) = 0 \ \forall x \in X\}$. By Hilbert's Nullstellensatz, $I(X)$ is the radical of the ideal $(f_1, \dots, f_r) : (g)$. The *coordinate ring* $A = k[X]$ of X is the localization $(R/I(X))_g$. Such a ring A will be called a (reduced) *affine k -algebra*.

In general, a variety is not affine. However, if $X = Y \setminus \mathcal{Z}(g_1, \dots, g_r)$ with affine Y , then X is covered by the affine open subsets $U_i := X \setminus \mathcal{Z}(g_i) = Y \setminus \mathcal{Z}(g_i)$, $1 \leq i \leq r$.

The *dimension* $\dim X$ is the *Krull dimension* of $X \subseteq \mathbb{A}^n$ in the Zariski topology. In the case $\dim X = n - 1$ we call X a *hypersurface*. The *degree* $\deg X$ of an irreducible variety $X \subseteq \mathbb{A}^n$ of dimension m is defined as the maximal cardinality of $X \cap L$ over all affine linear subspaces $L \subseteq \mathbb{A}^n$ of dimension $n - m$ [29, §5A]. We define the (cumulative) degree $\deg X$ of a reducible variety X to be the sum of the degrees of *all* irreducible components of X . It follows essentially from Bézout's Theorem that if $X \subseteq \mathbb{A}^n$ is closed and defined by polynomials of degree $\leq d$, then $\deg X \leq d^n$ [8].

An important ingredient of our work is the effective Nullstellensatz, which was first proved in [6, 25]. A recent version of Jelonek [24] generalizes the statement to polynomials on general affine varieties.

Theorem 2.1. *Let $X \subseteq \mathbb{A}^n$ be a closed subvariety of dimension m and degree D , and let $g_1, \dots, g_t \in k[X]$ be polynomials of degree at most $d \geq 1$ without common zeros in X . Then there exist polynomials $h_i \in k[X]$ with $1 = \sum_i h_i g_i$ and*

$$\deg(h_i g_i) \leq \begin{cases} Dd^t & \text{if } t \leq m, \\ Dd^m & \text{if } t > m, \ d \geq 3, \ m \geq n-1, \\ 2Dd^m - 1 & \text{else.} \end{cases}$$

Proof. The first and the last case is due to [24]. The slightly better bound in the middle case follows from [25]. \square

There is another version of the effective Nullstellensatz for arbitrary ideals due to Kollár [26]. We note a variant which follows easily from Theorem 6.2 in that paper.

Theorem 2.2. *Let $X_1, \dots, X_t \subseteq \mathbb{A}^n$ be closed varieties with $X_1 \cap \dots \cap X_t = \emptyset$. Then there exist $f_i \in I(X_i)$ such that*

$$\sum_i f_i = 1 \quad \text{and} \quad \deg f_i \leq (n+1) \prod_i \deg X_i.$$

2.2 Completions

We will also use complete rings, which we get from affine rings by the process of completion [13, Chapter 7]. Let A be an affine algebra and I an ideal in A . The *completion* $\hat{A} = \hat{A}_I$ of A with respect to I is defined as the inverse limit of the factor rings A/I^ν , $\nu \geq 0$. There is a canonical map $A \rightarrow \hat{A}$, whose kernel is $\bigcap_\nu I^\nu$, thus it is injective in our case. Alternatively, if $I = (f_1, \dots, f_r)$, one can define \hat{A} as $A[[T_1, \dots, T_r]]/(T_1 - f_1, \dots, T_r - f_r)$, so its elements are power series in f_1, \dots, f_r [13, Exercise 7.11]. For instance, if $A = B[T]$ and $I = (T)$, then $\hat{A} = B[[T]]$ is the ring of formal power series in T with coefficients in B .

2.3 Coherent Sheaves and Differential Forms

Let X be an affine variety. Then every $k[X]$ -module M gives rise to a sheaf \widetilde{M} on X such that, on a principal open subset $U = X \setminus \mathcal{Z}(g)$, the sections of \widetilde{M} are given by $\Gamma(U, \widetilde{M}) = M_g$, the localization of M at g [22, Proposition II.5.1]. Moreover, $\Gamma(U, \widetilde{M})$ is a $k[X]_g$ -module which is compatible with restrictions. A sheaf \mathcal{F} on X is called *coherent* iff $\mathcal{F} = \widetilde{M}$ with a finitely generated $k[X]$ -module M . An important example is the structure sheaf $\mathcal{O}_X = \widetilde{k[X]}$.

More generally, a sheaf \mathcal{F} on a locally closed set X is called *coherent* iff X can be covered by affine open subsets U_i such that all the restrictions $\mathcal{F}|_{U_i}$ are

coherent. In particular, if \mathcal{F} is a coherent sheaf on Y and $X \subseteq Y$ is an open subset, then $\mathcal{F}|_X$ is a coherent sheaf on X .

Now let A be a k -algebra (commutative, with 1). The module of *Kähler differentials* $\Omega_A := \Omega_{A/k}$ is defined as the A -module generated by symbols df for all $f \in A$, modulo the relations of Leibniz' rule and k -linearity for the *universal derivation* $d: A \rightarrow \Omega_A$. For instance, for the polynomial ring R , the module Ω_R is free with basis dX_1, \dots, dX_n , and the universal derivation is given by $df = \sum_i \frac{\partial f}{\partial X_i} dX_i$. Now let $\Omega_A^p := \bigwedge^p \Omega_A$ be the p -th exterior power of the A -module Ω_A . We define the *exterior differential* $d: \Omega_A^p \rightarrow \Omega_A^{p+1}$ by setting $d(fdg_1 \wedge \dots \wedge dg_p) := df \wedge g_1 \wedge \dots \wedge dg_p$ for all $f, g_i \in A$. It is easy to check that d satisfies the graded Leibniz' rule and $d \circ d = 0$. This way we obtain the *de Rham complex*

$$\Omega_A^\bullet: A = \Omega_A^0 \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_A^p \longrightarrow \dots$$

of the algebra A .

Now let Y be an affine variety with coordinate ring $A = k[Y]$. Then the *sheaf of regular differential p -forms* on Y is defined as $\Omega_Y^p := \widetilde{\Omega}_A^p$. The exterior differentials glue together to maps of sheaves $d: \Omega_Y^p \rightarrow \Omega_Y^{p+1}$. More generally, if $X \subseteq Y$ is an open subset of an affine variety Y , then $\Omega_X^p := \Omega_Y^p|_X$. This way we obtain the (*algebraic*) *de Rham complex*

$$\Omega_X^\bullet: \mathcal{O}_X = \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^p \longrightarrow \dots$$

of the variety X . We say that X is *smooth* at $x \in X$ iff the stalk $\Omega_{X,x}$ is a free $\mathcal{O}_{X,x}$ -module of rank m , where m is the maximal dimension of all irreducible components of X through x . In fact, in this case there is only one such component. We say that X is *smooth* iff it is smooth at all its points. Note that in this case $\Omega_X^p = 0$ for $p > m$.

The module of Kähler differentials of a complete ring may not be finitely generated (see, e.g., [13, Exercise 16.14]). In these cases we use the *universally finite* module of differentials, which is always finitely generated (see [27, §11–12]). Let A be an affine algebra, I an ideal in A , and \widehat{A} the completion of A with respect to I . The *completion* of Ω_A with respect to I is $\widehat{\Omega}_A = \widehat{A} \otimes_A \Omega_A$ and is called the *universally finite module of differentials of \widehat{A}* . There is a *universally finite derivation* $d: \widehat{A} \rightarrow \widehat{\Omega}_A$ which is continuous, i.e., it commutes with infinite sums. For instance, for an affine algebra B we have

$$\widehat{\Omega}_{B[[T]]} = B[[T]] \otimes_{B[[T]]} \Omega_{B[[T]]} = B[[T]]dT \oplus \Omega_B,$$

and the universally finite derivation is given by $df = \frac{\partial f}{\partial T}dT + d_B f$ for $f \in B[[T]]$, where $\frac{\partial f}{\partial T}$ denotes the formal partial derivative with respect to T , and $d_B f$ is coefficient-wise application of the differential of B [27, Example 12.7].

2.4 Sheaf and Local Cohomology

Let X be a variety. Formally, the q -th *sheaf cohomology* functor $H^q(X, \cdot)$ from the category of sheaves of abelian groups on X to the category of abelian groups

is defined as the q -th right derived functor of the global section functor $\Gamma(X, \cdot)$,

$$H^q(X, \cdot) = R^q\Gamma(X, \cdot).$$

Since this definition is computationally quite inconvenient, we prefer a different description. Let \mathcal{F} be a coherent sheaf and $\mathcal{U} := \{U_i \mid 0 \leq i \leq t\}$ an affine open cover of X . The sheaf cohomology of \mathcal{F} can be computed as the Čech cohomology with respect to \mathcal{U} , which is defined as follows. For a set of indices $0 \leq i_0, \dots, i_q \leq t$ denote $U_{i_0 \dots i_q} := U_{i_0} \cap \dots \cap U_{i_q}$. We define the vector spaces

$$C^q := C^q(\mathcal{U}, \mathcal{F}) := \bigoplus_{0 \leq i_0 < \dots < i_q \leq t} \mathcal{F}(U_{i_0 \dots i_q})$$

and the linear maps $\delta^q: C^q \rightarrow C^{q+1}$,

$$\delta^q(\omega)_{i_0 \dots i_{q+1}} := \sum_{\nu=0}^{q+1} (-1)^\nu \omega_{i_0 \dots \widehat{i_\nu} \dots i_{q+1}}|_{U_{i_0 \dots i_{q+1}}}. \quad (2)$$

Then one checks that $\delta^{q+1} \circ \delta^q = 0$ for all $q \geq 0$, so that $(C^\bullet(\mathcal{U}, \mathcal{F}), \delta^\bullet)$ is a complex called the *Čech complex*. Its cohomology defines the cohomology $H^\bullet(\mathcal{U}, \mathcal{F})$, i.e.,

$$H^q(\mathcal{U}, \mathcal{F}) := \frac{\ker \delta^q}{\operatorname{im} \delta^{q-1}}, \quad q \geq 0.$$

By [22, Theorem III.4.5] there is a natural isomorphism

$$H^q(X, \mathcal{F}) \simeq H^q(\mathcal{U}, \mathcal{F}) \quad \text{for all } q \geq 0.$$

Generally, $H^0(X, \mathcal{F})$ is isomorphic to the space of global sections $\Gamma(X, \mathcal{F})$. Moreover, if X is affine, the higher cohomology of \mathcal{F} vanishes, i.e.,

$$H^q(X, \mathcal{F}) = 0 \quad \text{for } q > 0.$$

A relative variant of sheaf cohomology is local cohomology. Let X be a variety and $Y \subseteq X$ a closed subset of X . For a sheaf of abelian groups \mathcal{F} on X let $\Gamma_Y(X, \mathcal{F})$ be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections s whose support is in Y , i.e.,

$$\{s_x \neq 0\} \subseteq Y,$$

where s_x denotes the germ of s in the stalk \mathcal{F}_x . The q -th *local cohomology functor with supports in Y* from the category of sheaves of abelian groups on X to the category of abelian groups is defined as the q -th right derived functor of the functor $\Gamma_Y(X, \cdot)$,

$$H_Y^q(X, \cdot) = R^q\Gamma_Y(X, \cdot).$$

Note that, if X is irreducible and $Y \neq X$, then $\Gamma_Y(X, \mathcal{F}) = 0$ for all coherent sheaves \mathcal{F} on X . Nevertheless, the definition gives non-trivial local cohomology groups also in this case! There is an explicit description of local cohomology

in terms of Koszul cohomology, which can also be interpreted as Čech cohomology [19, Theorem 2.3]. We will use this interpretation in a special case in the proof of Lemma 5.5, see (11). What makes local cohomology particularly useful, is the following long exact sequence [19, Corollary 1.9]. Let $Y \subseteq X$ be closed and $U := X \setminus Y$. Then there is an exact sequence

$$\cdots \rightarrow H^{q-1}(X, \mathcal{F}) \rightarrow H^{q-1}(U, \mathcal{F}|_U) \rightarrow H_Y^q(X, \mathcal{F}) \rightarrow H^q(X, \mathcal{F}) \rightarrow \cdots.$$

2.5 Hypercohomology and de Rham Cohomology

Let X be a variety and consider a complex of coherent sheaves (\mathcal{F}^\bullet, d) on X with $\mathcal{F}^p = 0$ for $p < 0$. Then, for an affine open cover \mathcal{U} , the Čech complexes $C^\bullet(\mathcal{U}, \mathcal{F}^p)$ as defined in §2.4 fit together to the Čech double complex $C^{\bullet, \bullet} := C^{\bullet, \bullet}(\mathcal{U}, \mathcal{F}^\bullet)$ by setting

$$C^{p,q}(\mathcal{U}, \mathcal{F}^\bullet) = \bigoplus_{i_0 < \cdots < i_q} \mathcal{F}^p(U_{i_0 \cdots i_q}) \quad \text{for all } p, q \geq 0.$$

The two differentials are the one induced by the differential d of \mathcal{F} and the Čech differential δ^\bullet defined by (2). Define the total complex of $C^{\bullet, \bullet}$ by

$$\text{tot}^\ell(C^{\bullet, \bullet}) := \bigoplus_{p+q=\ell} C^{p,q}, \quad d^{\text{tot}} := \delta^q + (-1)^q d \text{ on } C^{p,q}.$$

Since the two differentials commute, one easily checks that $d^{\text{tot}} \circ d^{\text{tot}} = 0$. The *hypercohomology* of the complex of sheaves \mathcal{F}^\bullet is defined as the cohomology of the total complex

$$\mathbb{H}^\ell(X, \mathcal{F}^\bullet) := H^\ell(\text{tot}^\bullet(C^{\bullet, \bullet})) \quad \text{for } \ell \geq 0.$$

As for any double complex [28, §2.4], there are two spectral sequences

$$\begin{aligned} {}_I E_2^{p,q} &= H^p(H^q(X, \mathcal{F}^\bullet), d) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet) \quad \text{and} \\ {}_{II} E_2^{p,q} &= H^q(H^p(C^{\bullet, \bullet}, d), \delta^\bullet) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet). \end{aligned}$$

If X is affine, then $H^q(X, \mathcal{F}^\bullet) = 0$ for all $q > 0$. Consequently, the first spectral sequence implies that $\mathbb{H}^\bullet(X, \mathcal{F}^\bullet) \simeq H^\bullet(\Gamma(X, \mathcal{F}^\bullet))$.

The *algebraic de Rham cohomology* of a variety X is defined as the hypercohomology of the algebraic de Rham complex

$$H_{\text{dR}}^\bullet(X) := \mathbb{H}^\bullet(X, \Omega_X^\bullet).$$

The corresponding double complex is called *Čech-de Rham double complex*. In particular, if X is affine, then we have

$$H_{\text{dR}}^\bullet(X) = H^\bullet(\Omega_A^\bullet), \quad \text{where } A = k[X].$$

This in turn implies that the first term of the second spectral sequence of the Čech-de Rham double complex is

$${}_{II}E_1^{p,q} = \bigoplus_{i_0 < \dots < i_q} H_{\text{dR}}^p(U_{i_0 \dots i_q}) \Rightarrow H_{\text{dR}}^{p+q}(X),$$

even when X is not affine.

Fundamental for us is the result of [18] stating that if $k = \mathbb{C}$, then the de Rham cohomology $H_{\text{dR}}^\bullet(X)$ of a smooth variety X is naturally isomorphic to the singular cohomology of X .

2.6 Filtrations

Now let $A = R/I$ be an affine algebra corresponding to a closed variety $X \subseteq \mathbb{A}^n$. We are interested in degree bounds for the de Rham cohomology of X , so we introduce the following notation. For $f \in R$ we denote by \bar{f} its residue class in A . We set

$$\deg \bar{f} := \min\{\deg h \mid h \in R, \bar{h} = \bar{f}\},$$

where $\deg h$ denotes the total degree of h . We have the *filtration by degree*

$$k = D^0 A \subseteq \dots \subseteq D^d A \subseteq D^{d+1} A \subseteq \dots \subseteq A$$

on A given by

$$D^d A := \{f \in A \mid \deg f \leq d\},$$

which satisfies

$$D^d A \cdot D^e A \subseteq D^{d+e} A \quad \text{for all } d, e \in \mathbb{Z}.$$

For every non-zerodivisor $g \in A$ we define the degree in the localization A_g by

$$\deg \frac{f}{g^s} := \deg f - s \deg g, \quad f \in A. \quad (3)$$

One easily checks that this degree is well-defined. Note that it can be arbitrarily small, so it induces an unbounded filtration by degree $D^\bullet A_g$. On A_g we also define the notion of *order* with respect to g by

$$\text{ord}_g h := \min\{s \mid \exists f \in A: h = \frac{f}{g^s}\} \quad \text{for } h \in A_g.$$

Sometimes we drop the index g , if it is clear from the context. We define the *filtration by order*

$$A = P^0 A_g \subseteq \dots \subseteq P^s A_g \subseteq P^{s+1} A_g \subseteq \dots \subseteq A_g$$

by setting

$$P^s A_g := \frac{1}{g^s} A = \{h \in A_g \mid \text{ord}_g h \leq s\}.$$

We also consider *modules* M over A equipped with a filtration

$$\cdots \subseteq F^d M \subseteq F^{d+1} M \subseteq \cdots \subseteq M,$$

which we assume to be *compatible* with $D^\bullet A$, i.e., $D^d A \cdot F^e M \subseteq F^{d+e} M$ for all $d, e \in \mathbb{Z}$. Given such a filtration, one can define a degree by setting

$$\deg x := \min\{d \mid x \in F^d M\} \in \mathbb{Z} \quad \text{for } x \in M.$$

As in (3), it induces a filtration $F^\bullet M_g$ in the localization M_g of M with respect to a non-zero-divisor $g \in A$, which is compatible with $D^\bullet A_g$. Moreover, there is a natural filtration $P^\bullet M_g$ by the order with respect to g , which is compatible with the corresponding filtration $P^\bullet A_g$.

For a complex of A -modules C^\bullet , all equipped with a filtration $F^\bullet C^\bullet$, we define the induced filtration on the cohomology $H^p(C^\bullet)$ by setting

$$F^d H^p(C^\bullet) := \text{im}(F^d C^p \cap \ker d \rightarrow H^p(C^\bullet)), \quad (4)$$

where $d: C^p \rightarrow C^{p+1}$ denotes the differential of the complex.

In order to conveniently state our results, we introduce some notation. Let M be a k -vector space. If M is equipped with one increasing filtration $F^\bullet M$, then we can define

$$\deg(M) := \inf\{d \in \mathbb{Z} \mid F^d M = M\}.$$

Note that $\deg(M) = \infty$ if no such d exists.

However, we also deal with simultaneous bounds on degrees and orders, so we need a more general concept. Assume that M is equipped with a *double filtration* $F^{\bullet, \bullet} M$, i.e., vector subspaces $F^{s,d} M \subseteq M$ for $s, d \in \mathbb{Z}$ such that $F^{s,d} M \subseteq F^{s+1,d} M \cap F^{s,d+1} M$ for all $s, d \in \mathbb{Z}$. Then we define the set

$$B(F^{\bullet, \bullet} M) := \{(s, d) \in \mathbb{Z}^2 \mid F^{s,d} M = M\}. \quad (5)$$

If the double filtration is understood from the context, we also simply write $B(M)$. Note that with $(s, d) \in B(F^{\bullet, \bullet} M)$ we have $(s, d) + \mathbb{N}^2 \subseteq B(F^{\bullet, \bullet} M)$. If $F^\bullet M$ and $P^\bullet M$ are two increasing filtrations on M , then clearly $F^{s,d} M := P^s M \cap F^d M$ for $s, d \in \mathbb{Z}$ defines a double filtration, which we call the *standard double filtration* induced by $F^\bullet M$ and $P^\bullet M$. Similarly as in (4), a double filtration on a complex induces a double filtration on its cohomology. A double filtration on a double complex induces in a natural way a double filtration on its total complex.

Particularly important for us are the modules of Kähler differential forms. Now let $A = (R/I)_g$ be an affine algebra, and consider the A -module Ω_A^p . We define the two filtrations

$$D^d \Omega_A^p := \left\{ \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} dX_{i_1} \wedge \cdots \wedge dX_{i_p} \mid f_{i_1 \cdots i_p} \in D^{d-p} A \right\},$$

$$P^s \Omega_A^p := \left\{ \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} dX_{i_1} \wedge \cdots \wedge dX_{i_p} \mid f_{i_1 \cdots i_p} \in P^s A \right\},$$

and note that they are compatible with the respective filtrations on A . Notice that we have induced degree and order functions satisfying

$$\begin{aligned}\deg f dX_{i_1} \wedge \cdots \wedge dX_{i_p} &= \deg f + p, \\ \text{ord}_g f dX_{i_1} \wedge \cdots \wedge dX_{i_p} &= \text{ord}_g f, \quad f \in A.\end{aligned}$$

The differential $d: \Omega_A^p \rightarrow \Omega_A^{p+1}$ satisfies

$$d(D^d \Omega_A^p) \subseteq D^d \Omega_A^{p+1}, \quad d(P^s \Omega_A^p) \subseteq P^{s+1} \Omega_A^{p+1}.$$

Using these notations and conventions, we formulate an easy consequence of the GDD bound.

Lemma 2.3. *Let $f \in R$ and $U := \mathbb{A}^n \setminus \mathcal{Z}(f)$. Then, with respect to the standard double filtration induced by $D^\bullet \Omega_{R_f}^p$ and $P^\bullet \Omega_{R_f}^p$, we have*

$$p \cdot (1, 1) \in B(H_{\text{dR}}^p(U)).$$

Proof. Denote by \tilde{f} the generous homogenization $X_0^{d+1}f(X/X_0)$, where $d := \deg f$. Then we have $U = \mathbb{P}^n \setminus \mathcal{Z}(\tilde{f})$. As stated in §1.2, each cohomology class in $H_{\text{dR}}^p(U)$ is represented by a differential form $\tilde{\alpha}/\tilde{f}^p$, where $\tilde{\alpha}$ is a homogeneous p -form on \mathbb{A}^{n+1} of degree $\deg \tilde{\alpha} = p(d+1)$. Dehomogenizing yields a form $\omega = \alpha/f^p$ with $\deg \alpha \leq p(d+1)$, hence $\deg \omega \leq p$ and $\text{ord}_f \omega \leq p$. \square

3 Zeroth Cohomology

In this section we discuss old and new results about the zeroth de Rham cohomology of a closed variety $X \subseteq \mathbb{A}^n$. This is a somewhat special case, since $H_{\text{dR}}^0(X)$ characterizes the connected components of X even if X is singular.

We fix the notation for this section. Let $X = Z_1 \cup \cdots \cup Z_t$ be the decomposition of X into connected components, set $D := \deg D$ and $D_i := \deg Z_i$. By the results of [7, §3.1.2], there is a direct product decomposition of the coordinate ring

$$k[X] \simeq \prod_{i=1}^t k[Z_i],$$

which corresponds to a maximal complete set of pairwise orthogonal idempotents $e_1, \dots, e_t \in k[X]$. Here, e_i is a function which is 1 on Z_i and zero on $X \setminus Z_i$, and any such function can be used in the set of idempotents. Furthermore, we have proved in [7] that e_1, \dots, e_t is a basis of $H_{\text{dR}}^0(X)$ and that $\deg e_i \leq d^{\mathcal{O}(n^2)}$, if X is given by polynomials of degree $\leq d$. Due to Jelonek's version of the effective Nullstellensatz, we are able to improve this bound.

Proposition 3.1. *Let $X \subseteq \mathbb{A}^n$ be a closed variety of dimension m and degree D . Then X has $\dim H_{\text{dR}}^0(X)$ connected components, and*

$$\deg(H_{\text{dR}}^0(X)) \leq D^{m+1}.$$

Proof. The result is known for $m = 0$, so assume $m \geq 1$. We construct the idempotents e_i as follows. According to [7, Proposition 2.1], each Z_i can be defined by (many) polynomials $f_{i\nu}$ of degree at most D_i . For $i \neq j$ we have $Z_i \cap Z_j = \emptyset$, hence the $f_{j\nu}$ have no common zero in Z_i . By Theorem 2.1 there exist polynomials g_ν with $\deg(g_\nu f_{j\nu}) \leq 2D_i D_j^m$ such that

$$\varphi_{ij} := \sum_{\nu} g_\nu f_{j\nu} = 1 \quad \text{on } Z_i.$$

It is easy to see that the desired idempotents can be defined as

$$e_i := \prod_{j < i} \varphi_{ij} \cdot \prod_{j > i} \varphi_{ij}.$$

Their degrees satisfy

$$\deg e_i \leq 2D_i \sum_{j \neq i} D_j^m \leq 2D_i \left(\sum_{j \neq i} D_j \right)^m = 2D_i (D - D_i)^m.$$

A small curve discussion shows that the last expression, as a function of D_i , is maximal for $D_i = \frac{D}{m+1}$, which implies $\deg e_i \leq \frac{2}{m+1} \left(\frac{m}{m+1} \right)^m D^{m+1} \leq D^{m+1}$ for $m \geq 1$. \square

This result gives a very good bound for small dimensions, say for curves. However, Kollár's effective Nullstellensatz for arbitrary ideals implies a bound which is quadratic in the degree and therefore gives better results for larger dimensions.

Proposition 3.2. *Let $X \subseteq \mathbb{A}^n$ be a closed variety of degree D . Then*

$$\deg(H_{\text{dR}}^0(X)) \leq \frac{n+1}{4} D^2. \quad (6)$$

Proof. By Theorem 2.2 there exist polynomials $\varphi_{ij} \in I(Z_i)$ and $\psi_{ij} \in I(Z_j)$ for $i \neq j$ such that

$$\deg(\varphi_{ij}), \deg(\psi_{ij}) \leq (n+1)D_i D_j \quad \text{and} \quad \varphi_{ij} + \psi_{ij} = 1.$$

Now the desired idempotents can be defined as

$$e_i := \prod_{j < i} \varphi_{ji} \cdot \prod_{j > i} \psi_{ij}.$$

Their degrees satisfy

$$\deg e_i \leq (n+1)D_i \sum_{j \neq i} D_j = (n+1)D_i (D - D_i) \leq (n+1) \left(\frac{D}{2} \right)^2. \quad \square$$

Remark 3.3. In [35] we have proved that for a hypersurface X the factor $n+1$ in (6) can be dropped.

Example 3.4. This example shows that the bound (6) is sharp up to the factor $n + 1$. It is derived from Example 2.3 of [25], which goes back to Masser, Philippon, and Brownawell [6].

Let $d \geq 1$. Consider the polynomials

$$f_1 := X_1, \quad f_2 := X_2 X_3^{d-1} - 1, \quad f_3 := X_1 X_3^{d-1} - X_2^d,$$

and set $Z_1 := \mathcal{Z}(f_1, f_2)$ and $Z_2 := \mathcal{Z}(f_3)$ in \mathbb{A}^3 . Clearly, both Z_1 and Z_2 are smooth irreducible varieties of degree d that do not intersect. Consequently, they are the connected components of $X := Z_1 \cup Z_2$, and $D := \deg X = 2d$. Now consider the projective closure $\overline{X} = \overline{Z_1} \cup \overline{Z_2}$. Let $F_i \in k[X_0, \dots, X_3]$ denote the homogenization of f_i , i.e.,

$$F_1 = X_1, \quad F_2 := X_2 X_3^{d-1} - X_0^d, \quad F_3 = X_1 X_3^{d-1} - X_2^d.$$

One easily checks that $I(\overline{Z_1}) = (F_1, F_2)$ and $I(\overline{Z_2}) = (F_3)$. Now let $e_1, e_2 \in k[X_1, X_2, X_3]$ denote the idempotents of Z_1 and Z_2 , $\delta := \max\{\deg e_1, \deg e_2\}$, and E_1, E_2 their homogenizations w.r.t. degree δ , i.e., $E_i = X_0^\delta e_i(X/X_0)$. Then we have

$$E_1 + E_2 = X_0^\delta \text{ on } \overline{X}, \quad E_1 = 0 \text{ on } \overline{Z_2}, \quad E_2 = 0 \text{ on } \overline{Z_1},$$

hence $X_0^\delta \in I(\overline{Z_1}) + I(\overline{Z_2})$. It follows that its image in

$$k[X_0, \dots, X_3]/(F_1, F_2, F_3, X_3 - 1) \simeq k[X_0]/(X_0^{d^2})$$

is also zero, so that

$$\max\{\deg e_1, \deg e_2\} = \delta \geq d^2 = D^2/4.$$

4 Some Reductions

Our aim is to show that it is sufficient to prove the claimed bounds for certain irreducible hypersurfaces. In this section we first reduce to the irreducible case, and then effectively construct a cover of X by affine open patches which are isomorphic to hypersurfaces. In the next section, we will show how degree bounds on those patches yield a bound for $H_{\text{dR}}^\bullet(X)$.

The reduction to the irreducible case follows from the effective characterization of the connected components discussed in §3.

Corollary 4.1. *If $X \subseteq \mathbb{A}^n$ is a smooth closed variety of degree D with irreducible components Z_i and $N := \min\{\frac{n+1}{4}D^2, D^{m+1}\}$, then*

$$\deg(H_{\text{dR}}^\bullet(X)) \leq \max_i \{\deg(H_{\text{dR}}^\bullet(Z_i))\} + N.$$

Proof. Since the irreducible components of X coincide with its connected components, the restrictions of differential forms to the Z_i induce an isomorphism

$$H_{\text{dR}}^\bullet(X) \simeq \bigoplus_i H_{\text{dR}}^\bullet(Z_i). \quad (7)$$

Let $e_i \in k[X]$ denote the idempotent corresponding to Z_i . Since e_i is 1 on Z_i and 0 on $X \setminus Z_i$, the embedding $H_{\text{dR}}^\bullet(Z_i) \hookrightarrow H_{\text{dR}}^\bullet(X)$ is induced by the map $\omega \mapsto e_i \omega$, so that Propositions 3.1 and 3.2 imply the claim. \square

It is well known that each irreducible variety X is birational to a hypersurface Y . By Zariski's Main Theorem, there exist open dense subsets $U \subseteq X$ and $V \subseteq Y$ which are isomorphic. Now we make this construction effective and obtain the result that degree and order bounds for $H_{\text{dR}}^\bullet(V)$ imply such bounds for $H_{\text{dR}}^\bullet(U)$.

The following lemma essentially consists of the construction of a *geometric resolution* [14, 31].

Lemma 4.2. *Let $X \subseteq \mathbb{A}^n$ be a closed irreducible subvariety of dimension $m < n$ and $\Xi \subseteq X$ be a finite subset. Then there exists a linear coordinate transformation after which X is in Noether normal position with algebraically independent variables X_1, \dots, X_m and a polynomial $f \in k[X_1, \dots, X_{m+1}]$ such that*

(i) *f is irreducible and monic in X_{m+1} ,*

(ii) *$\deg f \leq \deg X$,*

(iii) *$Y := \mathcal{Z}(f) \subseteq \mathbb{A}^{m+1}$ is the closure of the image of X under the projection $\pi: \mathbb{A}^n \rightarrow \mathbb{A}^{m+1}, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{m+1})$,*

(iv) *with $g := \frac{\partial f}{\partial X_{m+1}}$ the projection π restricts to an isomorphism*

$$U := X \setminus \mathcal{Z}(g) \xrightarrow{\simeq} V := Y \setminus \mathcal{Z}(g), \quad \text{and}$$

(v) *$g(\xi) \neq 0$ for all $\xi \in \Xi$.*

Proof. Let $A = R/I(X)$ be the coordinate ring of X . A linear coordinate transformation brings X into Noether normal position, i.e., we assume that the ring extension $k[X_1, \dots, X_m] \hookrightarrow A$ is integral. Let $r := n - m$. For $\lambda = (\lambda_1, \dots, \lambda_r) \in k^r$ let $u_\lambda := \lambda_1 X_{m+1} + \dots + \lambda_r X_n$. Then there exists a monic polynomial $f_\lambda \in k[X_1, \dots, X_m][T]$ with $f_\lambda(u_\lambda) = 0$ in A . Since A is a domain, we can assume f_λ to be irreducible. Let $\pi_\lambda: X \rightarrow \mathbb{A}^{m+1}$ be the map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_m, u_\lambda)$. Obviously, f_λ vanishes on the irreducible hypersurface $\pi_\lambda(X)$ of \mathbb{A}^{m+1} , so it is a reduced equation for it. It follows $\deg f_\lambda = \deg \pi_\lambda(X) \leq \deg X$ [23, Lemma 2]. Now we consider $\lambda_1, \dots, \lambda_r$ as variables and argue as above over the field $k_\lambda := \overline{k(\lambda_1, \dots, \lambda_r)}$. It is shown in [15, 32] that $f_\lambda \in k[\lambda, X_1, \dots, X_m, T]$. Now set $g_\lambda := \frac{\partial f_\lambda}{\partial T}$. Differentiating the equation $f_\lambda(u_\lambda) = 0$ with respect to λ_i , the chain rule yields

$$\frac{\partial f_\lambda}{\partial \lambda_i}(u_\lambda) + \frac{\partial f_\lambda}{\partial T}(u_\lambda) \frac{\partial u_\lambda}{\partial \lambda_i} = \frac{\partial f_\lambda}{\partial \lambda_i}(u_\lambda) + g_\lambda(u_\lambda) X_{m+i} = 0, \quad 1 \leq i \leq r.$$

We choose $\lambda \in k^r$ such that $g_\lambda(\xi_1, \dots, \xi_m, u_\lambda(\xi)) \neq 0$ for all $\xi = (\xi_1, \dots, \xi_n) \in \Xi$ and denote $f := f_\lambda$, $g := g_\lambda$, $w_i := \frac{\partial f_\lambda}{\partial \lambda_i}$. By another linear coordinate

transformation we can assume $u_\lambda = X_{m+1}$. It follows that the map

$$\mathcal{Z}(f) \setminus \mathcal{Z}(g) \rightarrow X \setminus \mathcal{Z}(g), \quad x = (x_1, \dots, x_{m+1}) \mapsto \left(x, -\frac{w_2(x)}{g(x)}, \dots, -\frac{w_r(x)}{g(x)} \right)$$

is an inverse of the projection, which concludes the proof. \square

Remark 4.3. If $X \subseteq \mathbb{A}^n$ is smooth, then V from Lemma 4.2 is also smooth.

Corollary 4.4. *In the situation of Lemma 4.2, consider the open subsets $U' := U \setminus \mathcal{Z}(h) \subseteq X$, where $h \in k[X]$, and $V' := \pi(U') \subseteq Y$. Then there exists $H \in k[Y]$ with $\deg H \leq \deg h \deg X$ such that $k[V'] = k[Y]_{gH}$. Moreover, the isomorphism $k[Y]_{gH} = k[V'] \xrightarrow{\sim} k[U'] = k[X]_{gh}$ maps any element of degree d and order s to an element of degree $\leq d$ and order $\leq s(\deg h + 1)$.*

Proof. By the proof of Lemma 4.2, there exist $w_i \in k[X_1, \dots, X_{m+1}]$ of degree $\leq \deg f$ such that

$$X_{m+i} = -w_i/g \quad \text{in } k[X]_g \quad \text{for } 1 \leq i \leq r = n - m.$$

Thus, the isomorphism $k[X]_g \xrightarrow{\sim} k[Y]_g$ maps h to the rational function

$$\tilde{h} := h(X_1, \dots, X_m, -\frac{w_1}{g}, \dots, -\frac{w_r}{g}) = H(X_1, \dots, X_{m+1})/g^{\deg h},$$

where $H \in k[X_1, \dots, X_{m+1}]$ with

$$\deg H = \deg \tilde{h} + \deg h \deg g \leq \deg h(1 + \deg g) \leq \deg h \deg X.$$

It follows $k[V'] = k[Y]_{gH}$, and this isomorphism identifies H with $g^{\deg h} \tilde{h}$, which implies the claim. \square

Corollary 4.5. *Let $X \subseteq \mathbb{A}^n$ be a closed irreducible variety of dimension m . Then there exist principal open subsets U_0, \dots, U_m covering X , such that each U_i is isomorphic to an open subset of a hypersurface described as in Lemma 4.2.*

Proof. We apply Lemma 4.2 successively to construct the U_i . We start with an arbitrary one-point set $\Xi_0 = \{\xi\} \subseteq X$ to obtain U_0 . Having constructed U_0, \dots, U_{i-1} , let Ξ_i contain a point from each irreducible component of $X \setminus \bigcup_{0 \leq j < i} U_j$, so that one obtains U_i with

$$\dim \left(X \setminus \bigcup_{0 \leq j < i} U_j \right) > \dim \left(X \setminus \bigcup_{0 \leq j \leq i} U_j \right).$$

It follows that $\dim X \setminus \bigcup_{0 \leq j \leq m} U_j = -1$, hence $\bigcup_{i=0}^m U_i = X$. \square

5 Effective Čech and Hypercohomology

In this section we show how the cohomologies of the open patches of Corollary 4.5 fit together to give effective descriptions of the cohomology of X . This is done via effective hypercohomology, for which we first need to make Čech cohomology effective.

So let $X \subseteq \mathbb{A}^n$ be a closed variety, and let \mathcal{F} be a coherent sheaf on X . Since X is affine, we have

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}), \quad H^q(X, \mathcal{F}) = 0 \quad \text{for } q > 0. \quad (8)$$

Now let $A := k[X]$ be the coordinate ring of X , and let M be a finitely generated A -module such that $\mathcal{F} = \widetilde{M}$. Furthermore, let $\mathcal{U} = \{U_i \mid 0 \leq i \leq t\}$ be a cover of X by principal open subsets $U_i = X \setminus \mathcal{Z}(g_i)$, where $g_i \in A$. By §2.4, the sheaf cohomology of \mathcal{F} can be computed as the Čech cohomology with respect to \mathcal{U} . Using §2.3, the Čech complex $C^\bullet = C^\bullet(\mathcal{U}, \mathcal{F})$ can be explicitly described as follows. For a set of indices $0 \leq i_0, \dots, i_q \leq t$ set $g_{i_0 \dots i_q} := g_{i_0} \cdots g_{i_q}$. Then

$$C^q = \bigoplus_{0 \leq i_0 < \dots < i_q \leq t} M_{g_{i_0 \dots i_q}}, \quad 0 \leq q \leq t.$$

The properties (8) mean that the augmented complex

$$0 \longrightarrow C^{-1} := M \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^{t-1}} C^t \longrightarrow 0$$

is exact, where δ^{-1} is induced by the restriction. Note that δ^{-1} yields the isomorphism $M \simeq H^0(X, \mathcal{F})$.

The point of this section is to study degree and order bounds for a preimage under δ^\bullet . So we consider a filtration

$$\dots \subseteq F^d M \subseteq F^{d+1} M \subseteq \dots \subseteq M$$

on M which is compatible with $D^\bullet A$. Recall from §2.1 that for a non-zero divisor $g \in A$ it induces a filtration $F^\bullet M_g$, and there is a natural filtration $P^\bullet M_g$ by the order with respect to g . These filtrations extend to the spaces C^q by setting

$$F^d C^q := \bigoplus_{0 \leq i_0 < \dots < i_q \leq t} F^d M_{g_{i_0 \dots i_q}} \quad \text{and} \quad P^s C^q := \bigoplus_{0 \leq i_0 < \dots < i_q \leq t} P^s M_{g_{i_0 \dots i_q}}.$$

By definition we have $\delta^q(F^d C^q) \subseteq F^d C^{q+1}$ and $\delta^q(P^s C^q) \subseteq P^s C^{q+1}$.

Lemma 5.1. *Let $\deg X = D$ and $\deg g_i \leq d_1$. For all $0 \leq q \leq t$ and each $\omega \in P^s C^q \cap F^d C^q \cap \ker \delta^q$ there exists $\eta \in P^s C^{q-1} \cap F^{d+N} C^{q-1}$ with $N := 2D(sd_1)^m$ such that $\omega = \delta^{q-1}(\eta)$.*

Proof. For $\omega = (\omega_{i_0 \dots i_q})_{i_0 < \dots < i_q} \in F^d C^q \cap P^s C^q$ we write $\omega_{i_0 \dots i_q} = \frac{\alpha_{i_0 \dots i_q}}{g_{i_0 \dots i_q}^s}$, where $\alpha_{i_0 \dots i_q} \in M$ with $\deg \alpha_{i_0 \dots i_q} \leq d + s \deg g_{i_0 \dots i_q}$. The assumption $\delta^q(\omega) = 0$ implies

$$\sum_{\nu} (-1)^\nu g_{i_\nu}^s \alpha_{i_0 \dots \widehat{i_\nu} \dots i_{q+1}} = 0 \quad \text{for all } 0 \leq i_0 < \dots < i_{q+1} \leq t. \quad (9)$$

Since \mathcal{U} is a cover of X , the g_0^s, \dots, g_t^s have no common zero. Thus, by Theorem 2.1 there are $h_0, \dots, h_t \in A$ with $\sum_i h_i g_i^s = 1$ and $\deg(h_i g_i^s) \leq N$. Now we let $\eta = (\eta_{i_0 \dots i_{q-1}}) \in C^{q-1}$ with

$$\eta_{i_0 \dots i_{q-1}} := \frac{1}{g_{i_0 \dots i_{q-1}}^s} \sum_{i=0}^t h_i \alpha_{i, i_0 \dots i_{q-1}} \quad \text{for all } 0 \leq i_0 < \dots < i_{q-1} \leq t.$$

Here we define $\alpha_{i, i_0 \dots i_{q-1}}$ to be zero, if $i \in \{i_0 \dots i_{q-1}\}$, and $\varepsilon \alpha_{j_0 \dots j_q}$ otherwise, where $\{i_0, \dots, i_{q-1}, i\} = \{j_0 < \dots < j_q\}$ and ε is the sign of the permutation sorting (i_0, \dots, i_{q-1}, i) . Obviously, we have $\text{ord} \eta_{i_0 \dots i_{q-1}} \leq s$. To bound its degree, note that for all i

$$\deg(h_i \alpha_{i, i_0 \dots i_{q-1}}) \leq N - s \deg g_i + d + s \deg g_{i, i_0 \dots i_{q-1}} = N + d + s \deg g_{i_0 \dots i_{q-1}},$$

which implies $\deg \eta \leq N + d$.

Finally, we have

$$\begin{aligned} \delta^{q-1}(\eta)_{i_0 \dots i_q} &= \sum_{\nu=0}^q (-1)^\nu \eta_{i_0 \dots \widehat{i_\nu} \dots i_q} |U_{i_0 \dots i_q} \\ &= \sum_{\nu=0}^q (-1)^\nu \frac{1}{g_{i_0 \dots \widehat{i_\nu} \dots i_q}^s} \sum_{i=0}^t h_i \alpha_{i, i_0 \dots \widehat{i_\nu} \dots i_q} \\ &= \frac{1}{g_{i_0 \dots i_q}^s} \sum_{i=0}^t h_i \sum_{\nu=0}^q (-1)^\nu g_{i_\nu}^s \alpha_{i, i_0 \dots \widehat{i_\nu} \dots i_q} \\ &\stackrel{(9)}{=} \frac{1}{g_{i_0 \dots i_q}^s} \sum_{i=0}^t h_i g_i^s \alpha_{i_0 \dots i_q} \\ &= \omega_{i_0 \dots i_q}, \end{aligned}$$

hence $\delta^{q-1}(\eta) = \omega$. □

Next we will bound the degree and order of the hypercohomology of a complex of coherent sheaves, given that we have degree and order bounds for the cohomologies of the open patches. This can be formulated in a general setting.

Lemma 5.2. *Let $C^{\bullet, \bullet}$ be a first quadrant double complex, where each $C^{p, q}$ is equipped with a double filtration $F^{\bullet, \bullet} C^{p, q}$. Then we have*

$$\bigcap_{p+q=\ell} B(F^{\bullet, \bullet} H^p(C^{\bullet, q})) \subseteq B(H^\ell(\text{tot}^\bullet(C^{\bullet, \bullet}))) \quad \text{for all } \ell \geq 0.$$

Proof. The second spectral sequence ${}_{II}E_r$ of the double complex $C^{\bullet, \bullet}$ has first term

$${}_{II}E_1^{p, q} = H^p(C^{\bullet, q}).$$

Note that the double filtration on $C^{\bullet,\bullet}$ induces a double filtration on the spectral sequence. Each term of the spectral sequence is a cohomology of the previous one realized as a subquotient, thus

$$B(F^{\bullet,\bullet} {}_{II}E_r^{p,q}) \subseteq B(F^{\bullet,\bullet} {}_{II}E_{r+1}^{p,q}) \quad \text{for all } r \geq 1.$$

Moreover, since the double complex is bounded, the spectral sequence collapses at a finite level, i.e., there exists r such that ${}_{II}E_r = {}_{II}E_\infty$. By convergence we have

$$H^\ell(\text{tot}^\bullet(C^{\bullet,\bullet})) \simeq \bigoplus_{p+q=\ell} {}_{II}E_\infty^{p,q}.$$

Finally, since this isomorphism respects the double filtrations $F^{\bullet,\bullet}C^{p,q}$, we conclude

$$B(H^\ell(\text{tot}^\bullet(C^{\bullet,\bullet}))) = \bigcap_{p+q=\ell} B(F^{\bullet,\bullet} {}_{II}E_\infty^{p,q}),$$

which implies the claim. \square

Now consider a bounded complex of coherent sheaves

$$\mathcal{F}^\bullet : 0 \longrightarrow \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{F}^u \longrightarrow 0$$

on X , and let M^p be finitely generated A -modules with $\mathcal{F}^p = \widetilde{M}^p$. Furthermore, let $\mathcal{U} = \{U_i \mid 0 \leq i \leq t\}$ be a cover of X by principal open subsets $U_i = X \setminus \mathcal{Z}(g_i)$ as above. Recall from §2.5 that the hypercohomology $\mathbb{H}^\ell(X, \mathcal{F}^\bullet)$ can be computed in two ways, namely as the cohomology of the total complex $\text{tot}^\bullet(C^{\bullet,\bullet}(\mathcal{U}, \mathcal{F}))$, and as the cohomology of the complex of global sections M^\bullet . Of course, the latter is much simpler, which is usually taken as an argument that one does not need hypercohomology in the affine setting. However, we want to go through these arguments to bound degrees.

Now assume that we have filtrations

$$\dots \subseteq F^d M^p \subseteq F^{d+1} M^p \subseteq \dots \subseteq M^p,$$

such that the differential d is of degree 0, i.e., $d(F^d M^p) \subseteq F^d M^{p+1}$. As above, this and the filtration by order naturally carry over to the Čech double complex and to its total complex.

Theorem 5.3. *Let $\deg X = D$ and $\deg g_i \leq d_1$. Then for all ℓ and all*

$$(s, d) \in \bigcap_{p+q=\ell} \bigcap_{i_0 < \dots < i_q} B(H^p(M_{g_{i_0} \dots g_{i_q}}^\bullet))$$

we have

$$\deg(H^\ell(M^\bullet)) \leq d + 2D(\ell + 1)(s + \ell)^m d_1^m.$$

Proof. Fix ℓ and set $N := 2D(s+\ell)^m d_1^m$. By Lemma 5.2 using the standard filtration $F^{s,d}C^{p,q} := P^s C^{p,q} \cap F^d C^{p,q}$, each cohomology class $c \in H^\ell(\text{tot}^\bullet(C^{\bullet,\bullet}))$ can be represented by an element $\omega = (\omega_{0,\ell}, \omega_{1,\ell-1}, \dots, \omega_{\ell,0})$, where

$$\omega_{p,q} \in F^{s,d}C^{p,q} \quad \text{for all } p+q = \ell.$$

Since $d^{\text{tot}}(\omega) = 0$, we have in particular $\delta^\ell(\omega_{0,\ell}) = 0$, so $\omega_{0,\ell}$ defines an element in $H^\ell(X, \mathcal{F}^0)$. By Lemma 5.1 there exists

$$\eta \in F^{s,d+N}C^{0,\ell-1}$$

with $\delta^{\ell-1}(\eta) = \omega_{0,\ell}$. Consider

$$\omega' := (0, \omega_{1,\ell-1} + (-1)^\ell d\eta, \omega_{2,\ell-2}, \dots, \omega_{\ell,0}).$$

Since $\omega - \omega' = (\omega_{0,\ell}, (-1)^{\ell-1} d\eta, 0, \dots, 0) = d^{\text{tot}}(\eta, 0, \dots, 0)$, ω and ω' define the same cohomology class c . Note that $\omega_{1,\ell-1} + (-1)^\ell d\eta \in F^{s+1,d+N}C^{1,\ell-1}$. Continuing this way, after ℓ steps we have a representative of c of the form $\omega'' = (0, \dots, 0, \omega''_{\ell,0})$, where

$$\omega''_{\ell,0} \in F^{s+\ell,d+\ell N}C^{\ell,0}.$$

The closedness of ω'' means that $\delta^0(\omega''_{\ell,0}) = 0$ and $d(\omega''_{\ell,0}) = 0$. Applying Lemma 5.1 once more yields $\alpha \in F^{d+(\ell+1)N}M^\ell$ with $\delta^{-1}(\alpha) = \omega''_{\ell,0}$. Moreover, $d(\alpha) = 0$, hence α defines a class in $H^\ell(M^\bullet)$, which is the image of c under the isomorphism $H^\ell(\text{tot}^\bullet(C^{\bullet,\bullet})) \simeq H^\ell(M^\bullet)$. \square

We close this section by discussing the case of an open subset of $X := \mathbb{A}^n$. Let $Z = \mathcal{Z}(f_0, \dots, f_r) \subseteq X$ be a closed subvariety and consider its complement $W := X \setminus Z$. If $r > 0$, then W is not affine, and as a result its de Rham cohomology $H_{\text{dR}}^\bullet(W)$ cannot be computed by global sections, so we have to use hypercohomology instead. The open subsets $U_i := X \setminus \mathcal{Z}(f_i)$, $0 \leq i \leq r$, form an open cover of W with corresponding Čech-de Rham double complex

$$C^{p,q} = \bigoplus_{i_0 < \dots < i_q} \Omega_W^p(U_{i_0 \dots i_q}) \quad \text{for all } p, q \geq 0,$$

where $U_{i_0 \dots i_q} = U_{i_0} \cap \dots \cap U_{i_q}$. The differentials $d: C^{p,q} \rightarrow C^{p+1,q}$ are induced by the exterior derivatives, and the differentials $\delta^q: C^{p,q} \rightarrow C^{p,q+1}$ are the Čech differentials. The de Rham cohomology of W is the cohomology of the total complex $\text{tot}^\bullet(C^{\bullet,\bullet})$, i.e.,

$$H_{\text{dR}}^\ell(W) = \mathbb{H}^\ell(W, \Omega_W^\bullet) = H^\ell(\text{tot}^\bullet(C^{\bullet,\bullet})).$$

Lemma 5.4. *Equip each $C^{p,q}$ with the shifted double filtration $S^{s,d}C^{p,q} := P^{s-q}C^{p,q} \cap F^{d-q}C^{p,q}$ for $s, d \in \mathbb{N}$. Then, with respect to the induced double filtration on the total complex, we have*

$$\ell \cdot (1, 1) \in B(S^{\bullet,\bullet}H_{\text{dR}}^\ell(W)).$$

Proof. By Lemma 2.3 we have with respect to the shifted double filtration

$$\ell \cdot (1, 1) \in B(S^{\bullet, \bullet} H_{\text{dR}}^p(U_{i_0 \dots i_q})) \quad \text{for all } p + q = \ell, \quad 0 \leq i_0 < \dots < i_q \leq r.$$

Applying Lemma 5.2 implies the claim. \square

In the case of a complete intersection we are able to give a more succinct description of $H_{\text{dR}}^{\bullet}(W)$ using sheaf cohomology as follows.

Lemma 5.5. *Let $\dim Z = n - r - 1$. Then, for $\ell > 0$, the projection map*

$$\text{tot}^{\ell}(C^{\bullet, \bullet}) = C^{\ell-r, r} \oplus \dots \oplus C^{\ell, 0} \twoheadrightarrow C^{\ell-r, r} = \Omega_{A_{f_0 \dots f_r}}^{\ell-r}$$

induces an isomorphism

$$H_{\text{dR}}^{\ell}(W) \xrightarrow{\simeq} H^{\ell-r}(H^r(W, \Omega_W^{\bullet})). \quad (10)$$

Proof. Let \mathcal{F} be a locally free sheaf on X . The long exact sequence of local cohomology is

$$\dots \rightarrow H^{q-1}(X, \mathcal{F}) \rightarrow H^{q-1}(W, \mathcal{F}|_W) \rightarrow H_Z^q(X, \mathcal{F}) \rightarrow H^q(X, \mathcal{F}) \rightarrow \dots$$

Since $H^{q-1}(X, \mathcal{F}) = 0$ for $q > 1$, we conclude

$$H^{q-1}(W, \mathcal{F}|_W) \simeq H_Z^q(X, \mathcal{F}) \quad \text{for } q > 1. \quad (11)$$

Since Z is a complete intersection of codimension $r + 1$, it follows from [19, Theorem 3.8], that $H_Z^q(X, \mathcal{F}) = 0$ for $q < r + 1$. Furthermore, since $W = X \setminus Z$ can be covered by $r + 1$ affine open subsets, Čech cohomology implies $H_Z^q(X, \mathcal{F}) = H^{q-1}(W, \mathcal{F}|_W) = 0$ for $q > r + 1$. Thus

$$H_Z^q(X, \mathcal{F}) = 0 \quad \text{for } q \neq r + 1. \quad (12)$$

The beginning of the long exact sequence together with (12) implies

$$H^0(W, \mathcal{F}|_W) = H^0(X, \mathcal{F}). \quad (13)$$

Now, the first spectral sequence of hypercohomology has first term

$${}_I E_1^{pq} = H^q(W, \Omega_W^p) \Rightarrow H_{\text{dR}}^{p+q}(W).$$

By (11), (12), and (13) we have

$${}_I E_1^{pq} = \begin{cases} H^0(X, \Omega_X^p) & \text{if } q = 0, \\ H^r(W, \Omega_W^p) & \text{if } q = r, \\ 0 & \text{otherwise.} \end{cases}$$

Since $H_{\text{dR}}^p(X) = H^p(H^0(X, \Omega_X^p)) = 0$ for $p > 0$, the second page ${}_I E_2$ has differential zero, hence ${}_I E_2 = {}_I E_{\infty}$, and we conclude

$$H_{\text{dR}}^{\ell}(W) = {}_I E_2^{\ell-r, r} = H^{\ell-r}(H^r(W, \Omega_W^{\bullet})) \quad \text{for } \ell > 0. \quad \square$$

Lemmas 5.4 and 5.5 imply

Corollary 5.6.

$$(\ell - r) \cdot (1, 1) \in B(H^{\ell-r}(H^r(W, \Omega_W^{\bullet}))).$$

6 Effective Gysin Sequence

The main tool in our proof is the Gysin sequence which is the following

Theorem 6.1. *Let X be a smooth irreducible variety and $Z \subseteq X$ a smooth closed equidimensional subvariety of codimension r . Then there is an exact sequence*

$$\cdots \rightarrow H_{\text{dR}}^p(X) \rightarrow H_{\text{dR}}^p(X \setminus Z) \xrightarrow{\text{Res}} H_{\text{dR}}^{p-2r+1}(Z) \rightarrow H_{\text{dR}}^{p+1}(X) \rightarrow \cdots$$

Let us first record an easy consequence of the Gysin sequence for the case that X is the affine space \mathbb{A}^n . Since $H_{\text{dR}}^p(X) = 0$ for $p > 0$, Theorem 6.1 implies

Corollary 6.2. *For a smooth closed equidimensional variety $Z \subseteq \mathbb{A}^n$ of codimension r the residue map*

$$\text{Res}: H_{\text{dR}}^p(\mathbb{A}^n \setminus Z) \xrightarrow{\cong} H_{\text{dR}}^{p-2r+1}(Z)$$

is an isomorphism for all $p > 0$.

An algebraic proof of Theorem 6.1 is given in [20]. Using this idea we will prove an effective version of it, i.e., we will describe the map Res explicitly on the level of differential forms, so that we can control its effect on their degrees.

However, our version only deals with the case of a smooth codimension 2 complete intersection of a very special type. In particular, we will study the following case. Denote $A := k[X_0, \dots, X_n]$, let $f, g \in R := k[X_1, \dots, X_n]$, and consider the regular sequence $f_0 := gX_0 - 1$, $f_1 := f$. Set $X := \mathbb{A}^{n+1}$, $Z := \mathcal{Z}(f_0, f_1)$ and denote its coordinate ring by $B := k[Z]$. Assume that f is irreducible and that $\frac{\partial f}{\partial X_n} \nmid g$. Note that these assumptions imply that Z is a smooth complete intersection in X of codimension 2, and its vanishing ideal is $I := I(Z) = (f_0, f_1)$, hence $B = A/(f_0, f_1)$.

Since in our case the complement $W := X \setminus Z$ is not affine, its de Rham cohomology $H_{\text{dR}}^\bullet(W)$ is described as in the last section. Recall that by Lemma 5.5 we have an isomorphism

$$H_{\text{dR}}^{p+1}(W) \xrightarrow{\cong} H^p(H^1(W, \Omega_W^\bullet)) = H^p\left(\frac{\Omega_{A_{f_0f_1}}^\bullet}{\Omega_{A_{f_0}}^\bullet + \Omega_{A_{f_1}}^\bullet}\right) \quad \text{for } p \geq 0. \quad (14)$$

The main result of this section is

Theorem 6.3. *Let $f_0, f_1 \in A$ be as above, and denote $d_0 := \deg f_0$, $d_1 := \deg f_1$. Then, under the identification (14), the map*

$$\text{Res}: H_{\text{dR}}^{p+1}(W) \rightarrow H_{\text{dR}}^{p-2}(Z) \quad \text{for } p \geq 0$$

is induced by a map $\Omega_{A_{f_0f_1}}^p \rightarrow \Omega_B^{p-2}$ which takes a p -form $\omega = \frac{\alpha}{(f_0f_1)^s}$ to a $(p-2)$ -form $\text{Res}(\omega)$ with

$$\deg \text{Res}(\omega) \leq (2d_0 - d_1 + 1)^{2s-1} \deg \alpha. \quad (15)$$

We fix the notations and assumptions of this theorem for the rest of the section. For its proof we will need the completion $\hat{A} = \hat{A}_I$ of A with respect to I . Recall from §2.2 that $\hat{A} \simeq A[[T_0, T_1]]/(T_0 - f_0, T_1 - f_1)$, so its elements are power series in f_0, f_1 . Note, however, that these power series are not unique. E.g., $f_0 \in A$ can be represented by the constant power series f_0 or by T_0 . The crucial result for us is a lemma of Grothendieck stating that there is an algebra isomorphism $B[[T_0, T_1]] \rightarrow \hat{A}$ (cf. [21, Lemma II,1.2]), which establishes a unique power series representation for the completion. We need to construct this isomorphism explicitly in order to bound degrees. The technical construction is in the following statement, which is a consequence of the fact that B is a *formally smooth* k -algebra [17, Definition 19.3.1].

For a tuple $x = (x_1, \dots, x_n)$ over an affine algebra C we write $\deg(x) := \max_j \deg(x_j)$ and use an analogous notation for the order. If $\psi: C \rightarrow D$ is a homomorphism, we write $\psi(x) := (\psi(x_1), \dots, \psi(x_n))$. For $x \in C$ we denote by \overline{x} its image in any factor algebra of C .

Lemma 6.4. *Let $N \in \mathbb{N}$ and $\psi: B \rightarrow A/I^N$ be an algebra homomorphism that lifts the identity $B \rightarrow B$, i.e., the composition $B \rightarrow A/I^N \rightarrow B$ is the identity. Then ψ can be lifted to an algebra homomorphism $\tilde{\psi}: B \rightarrow A/I^{N+1}$, i.e., the diagram*

$$\begin{array}{ccc} & & A/I^{N+1} \\ & \nearrow \tilde{\psi} & \downarrow \pi \\ B & \xrightarrow{\psi} & A/I^N \end{array}$$

commutes.

Proof. Since the k -algebra B is generated by $\overline{X}_0, \dots, \overline{X}_n$, it is sufficient to define $\tilde{\psi}$ on these elements. Choose $Y_0, \dots, Y_n \in A$ such that $\psi(\overline{X}_i) = \overline{Y}_i$ in A/I^N for all $0 \leq i \leq n$. Our aim is to define

$$\tilde{\psi}(\overline{X}_i) := \overline{Y_i + \sum_{\mu+\nu=N} a_{\mu\nu}^{(i)} f_0^\mu f_1^\nu} \quad \text{for } 0 \leq i \leq n, \quad (16)$$

with suitably chosen $a_{\mu\nu}^{(i)} \in A$. Then it is clear that $\pi \circ \tilde{\psi} = \psi$. It remains to show that one can define $\tilde{\psi}$ unambiguously by (16). This means that we have to find $a_{\mu\nu}^{(i)}$ such that f_0, f_1 are mapped to zero in A/I^{N+1} . Set $Y := (Y_0, \dots, Y_n)$, $a_{\mu\nu} := (a_{\mu\nu}^{(0)}, \dots, a_{\mu\nu}^{(n)})$, and look at the first condition

$$f_0 \mapsto \overline{f_0(Y + \sum_{\mu+\nu=N} a_{\mu\nu} f_0^\mu f_1^\nu)} = 0 \quad \text{in } A/I^{N+1}. \quad (17)$$

By the Taylor formula we have

$$f_0\left(Y + \sum_{\mu+\nu=N} a_{\mu\nu} f_0^\mu f_1^\nu\right) \equiv f_0(Y) + \sum_{i=0}^n \frac{\partial f}{\partial X_i}(Y) \sum_{\mu+\nu=N} a_{\mu\nu}^{(i)} f_0^\mu f_1^\nu \pmod{I^{N+1}}.$$

Since $\overline{f_0(Y)} = \psi(\overline{f_0}) = 0$ in A/I^N and $I^N = (f_0^N, f_0^{N-1}f_1, \dots, f_0f_1^{N-1}, f_1^N)$, there exist $p_{\mu\nu} \in A$, $\mu + \nu = N$, such that $f_0(Y) = \sum_{\mu+\nu=N} p_{\mu\nu} f_0^\mu f_1^\nu$ in A . Furthermore, since $\overline{Y}_i = \overline{X}_i$ in B , condition (17) is satisfied if

$$p_{\mu\nu} + \sum_{i=0}^n \frac{\partial f_0}{\partial X_i} a_{\mu\nu}^{(i)} \equiv 0 \pmod{I}. \quad (18)$$

Similarly, there exist $q_{\mu\nu} \in A$, $\mu + \nu = N$, such that $f_1(Y) = \sum_{\mu+\nu=N} q_{\mu\nu} f_0^\mu f_1^\nu$ in A , and f_1 is mapped to zero if

$$q_{\mu\nu} + \sum_{i=0}^n \frac{\partial f_1}{\partial X_i} a_{\mu\nu}^{(i)} \equiv 0 \pmod{I}. \quad (19)$$

In order to get very efficient degree bounds, we use the special form of the defining equations f_0, f_1 . In particular, recall that there exists $h \in A$ such that $\frac{\partial f_0}{\partial X_0} = g = h \frac{\partial f_1}{\partial X_n}$. Also, note that $\frac{\partial f_0}{\partial X_0} = 0$. This allows us to solve the linear system of equations over B consisting of (18) and (19) as follows:

$$\begin{aligned} a_{\mu\nu}^{(n)} &:= -q_{\mu\nu} h X_0, \\ a_{\mu\nu}^{(i)} &:= 0 \quad \text{for } 1 \leq i < n, \\ a_{\mu\nu}^{(0)} &:= -X_0(p_{\mu\nu} + \frac{\partial f_0}{\partial X_n} a_{\mu\nu}^{(n)}). \end{aligned} \quad (20)$$

We check that these settings actually solve the system:

$$q_{\mu\nu} + \sum_{i=0}^n \frac{\partial f_1}{\partial X_i} a_{\mu\nu}^{(i)} \equiv q_{\mu\nu} - \frac{\partial f_1}{\partial X_n} q_{\mu\nu} h X_0 \equiv -q_{\mu\nu} f_0 \equiv 0 \pmod{I}. \quad (21)$$

Moreover, we have

$$\begin{aligned} p_{\mu\nu} + \sum_{i=0}^n \frac{\partial f_0}{\partial X_i} a_{\mu\nu}^{(i)} &\equiv p_{\mu\nu} - \frac{\partial f_0}{\partial X_0} X_0(p_{\mu\nu} + \frac{\partial f_0}{\partial X_n} a_{\mu\nu}^{(n)}) + \frac{\partial f_0}{\partial X_n} a_{\mu\nu}^{(n)} \\ &\equiv -(p_{\mu\nu} + \frac{\partial f_0}{\partial X_n} a_{\mu\nu}^{(n)}) f_0 \equiv 0 \pmod{I}, \end{aligned} \quad (22)$$

which concludes the proof. \square

Corollary 6.5. *There exists an embedding $\psi: B \hookrightarrow \hat{A}$ such that $\psi(\overline{X}_i) = \sum_{\mu, \nu=0}^{\infty} a_{\mu\nu}^{(i)} f_0^\mu f_1^\nu$, where $a_{\mu\nu}^{(i)} \in A$ with*

$$\deg a_{\mu\nu}^{(i)} \leq 2d_0 - d_1 + 1 \quad \text{for all } \mu, \nu \in \mathbb{N}, 0 \leq i \leq n.$$

Proof. We start with $\psi_1 := \text{id}_B$ and apply Lemma 6.4 successively to construct the homomorphisms $\psi_N: B \rightarrow A/I^N$, $N \in \mathbb{N}$. Together they define a homomorphism $\psi: B \rightarrow \hat{A}$, which is clearly injective. To prove the degree bound,

denote by $Y^{(N)} = (Y_0^{(N)}, \dots, Y_n^{(N)}) \in A^{n+1}$ representatives of $\psi_N(\overline{X}_i) = \overline{Y_i^{(N)}}$ in A/I^N . Furthermore, let $p_{\mu\nu}^{(N)}, q_{\mu\nu}^{(N)} \in A$ with

$$f_0(Y^{(N)}) = \sum_{\mu+\nu=N} p_{\mu\nu}^{(N)} f_0^\mu f_1^\nu, \quad f_1(Y^{(N)}) = \sum_{\mu+\nu=N} q_{\mu\nu}^{(N)} f_0^\mu f_1^\nu.$$

Then, equation (21) shows that

$$\deg q_{\mu\nu}^{(N+1)} \leq \deg q_{\mu-1,\nu}^{(N)} \quad \text{for all } N \geq 1.$$

Since $q_{01}^{(1)} = 1$ and $q_{10}^{(1)} = 0$, we conclude inductively that

$$\deg q_{\mu\nu}^{(N)} \leq 0 \quad \text{for all } N \geq 1.$$

Moreover, by (22) we have

$$\deg p_{\mu\nu}^{(N+1)} \leq \max\{\deg p_{\mu-1,\nu}^{(N)}, d_0 - 1 + \deg h + 1\}.$$

Using $p_{01}^{(1)} = 0$ and $p_{10}^{(1)} = 1$, an induction proves

$$\deg p_{\mu\nu}^{(N)} \leq d_0 + \deg h = 2d_0 - d_1 \quad \text{for all } N \geq 1.$$

The solution (20) implies $\deg a_{\mu\nu}^{(n)} = d_0 - d_1 + 1$, and hence for all i

$$\deg a_{\mu\nu}^{(i)} \leq 1 + \max\{2d_0 - d_1, d_0 - 1 + d_0 - d_1 + 1\} = 2d_0 - d_1 + 1. \quad \square$$

For $a \in \widehat{A}$ we write $\deg_{\mu\nu} a \leq \delta_{\mu\nu}$, if there exists a representation $a = \sum_{\mu,\nu \geq 0} a_{\mu\nu} f_0^\mu f_1^\nu$ with $\deg a_{\mu\nu} \leq \delta_{\mu\nu}$ for all $\mu, \nu \in \mathbb{N}$. Denote $\xi_i := \psi(\overline{X}_i)$ and $\xi := (\xi_0, \dots, \xi_n)$. Then the degree bound of the previous corollary reads in this notation $\deg_{\mu\nu}(\xi) \leq 2d_0 - d_1 + 1 =: \gamma$.

Remark 6.6. A straight-forward induction with respect to the degree of p shows

$$\deg_{\mu\nu} p(\xi) \leq \gamma \deg p \quad \text{for all } p \in A. \quad (23)$$

Corollary 6.7. *For all $a = \sum_{\mu,\nu \geq 0} a_{\mu\nu} f_0^\mu f_1^\nu \in \widehat{A}$ there exist unique $b \in B$ and $c \in \widehat{I} = I\widehat{A}$ with $a = \psi(b) + c$. Furthermore, we have*

$$\deg b \leq \deg a_{00}, \quad (24)$$

and there exist $d, e \in \widehat{A}$ with $c = df_0 + ef_1$ and

$$\begin{aligned} \deg_{\mu\nu} d &\leq \max\{\deg a_{\mu+1,\nu}, \gamma \deg a_{00}\}, \\ \deg_{\mu\nu} e &\leq \max\{\deg a_{\mu,\nu+1}, \gamma \deg a_{00}\}. \end{aligned} \quad (25)$$

Proof. We have the exact sequence of \widehat{A} -modules

$$0 \longrightarrow \widehat{I} \longrightarrow \widehat{A} \xrightarrow{\pi} B \longrightarrow 0,$$

which splits by the homomorphism ψ . For this reason, $\widehat{A} \simeq B \oplus \widehat{I}$, and the existence and uniqueness of the claimed representation follows. Note that if $a = \psi(b) + c$, then $b = \pi(a) = \pi(a_{00})$. This implies (24).

Since ψ and π are k -algebra homomorphisms and a_{00} is a polynomial, we have $\psi(b) = \psi(\pi(a_{00})) = a_{00}(\xi)$, thus $c = a - a_{00}(\xi)$, and using (23) we conclude $\deg_{\mu,\nu} c \leq \max\{\deg a_{\mu\nu}, \gamma \deg a_{00}\}$, which yields (25). \square

Now we define the homomorphism

$$\widehat{\psi}: B[[T_0, T_1]] \rightarrow \widehat{A}, \quad \sum_{\mu, \nu \geq 0} b_{\mu\nu} T_0^\mu T_1^\nu \mapsto \sum_{\mu, \nu \geq 0} \psi(b_{\mu\nu}) f_0^\mu f_1^\nu.$$

Lemma 6.8. *The homomorphism $\widehat{\psi}$ is an isomorphism. For $a \in A$ we have*

$$\deg_{\mu\nu} \widehat{\psi}^{-1}(a) \leq \gamma^{\mu+\nu} \deg a \quad \text{for } \mu, \nu \geq 0.$$

Proof. We first prove that $\widehat{\psi}$ is injective. It is clear that $\psi_N: B \rightarrow A/I^N = \widehat{A}/\widehat{I}^N$ is injective for all $N \geq 1$. Note also, that by construction the diagram

$$\begin{array}{ccc} B & \xrightarrow{\psi} & \widehat{A} \\ & \searrow \psi_N & \downarrow \\ & & \widehat{A}/\widehat{I}^N \end{array}$$

commutes. We show that for all $N \geq 1$ and all $b_{\mu\nu} \in B$, $\mu + \nu = N - 1$, we have

$$\sum_{\mu+\nu=N-1} \psi(b_{\mu\nu}) f_0^\mu f_1^\nu \equiv 0 \pmod{\widehat{I}^N} \Rightarrow b_{\mu\nu} = 0 \text{ for all } \mu, \nu. \quad (26)$$

For $N = 1$ we have $0 \equiv \psi(b_{00}) \equiv \psi_1(b_{00}) \pmod{\widehat{I}}$, hence $b_{00} = 0$ by the injectivity of ψ_1 . Now assume that (26) is true for some $N \geq 1$, and assume that $\sum_{\mu+\nu=N} \psi(b_{\mu\nu}) f_0^\mu f_1^\nu \equiv 0 \pmod{\widehat{I}^{N+1}}$. Reducing mod (f_0) yields $\psi(b_{0,N}) f_1^N \equiv 0 \pmod{(f_0)}$. Since f_1 is a non-zerodivisor mod (f_0) , we conclude $\psi(b_{0,N}) \equiv 0 \pmod{(f_0)}$, thus $0 \equiv \psi(b_{0,N}) \equiv \psi_1(b_{0,N}) \pmod{\widehat{I}}$. Injectivity of ψ_1 implies $b_{0,N} = 0$. Now we write

$$\psi(b_{N,0}) f_0^N + \psi(b_{N-1,1}) f_0^{N-1} f_1 + \cdots + \psi(b_{1,N-1}) f_0 f_1^{N-1} = f_0 \cdot a,$$

where $a \in \widehat{A}$ has the form of the assumption in (26). Since $f_0 a \equiv 0 \pmod{\widehat{I}^{N+1}}$, there exist $g_{\lambda\eta} \in \widehat{A}$ with $f_0 a = \sum_{\lambda+\eta=N+1} g_{\lambda\eta} f_0^\lambda f_1^\eta$. Since f_1 is a non-zerodivisor mod (f_0) , we have $g_{0,N+1} \equiv 0 \pmod{(f_0)}$, and since f_0 is a non-zerodivisor, we infer $a \equiv 0 \pmod{\widehat{I}^N}$. The induction hypothesis implies $b_{\mu\nu} = 0$ for all μ, ν with $\mu + \nu = N$ and $\mu > 0$, which completes the proof of (26).

Now let $b = \sum_{\mu, \nu \geq 0} b_{\mu\nu} T_0^\mu T_1^\nu \in B[[T_0, T_1]]$ with $\widehat{\psi}(b) = 0$. We apply (26) inductively to conclude that $b = 0$.

To show surjectivity, let $a \in \widehat{A}$ and construct a preimage $\sum_{\mu, \nu} b_{\mu\nu} T_0^\mu T_1^\nu \in B[[T_0, T_1]]$ of a under $\widehat{\psi}$. We find the $b_{\mu\nu}$ successively by applying Corollary 6.7. Let $b_{00} \in B$ and $d_{10}, d_{01} \in \widehat{A}$ with $a = \psi(b_{00}) + d_{10}f_0 + d_{01}f_1$. Assume inductively, that for some $N \geq 1$ we have constructed $b_{\mu\nu} \in B$ for $\mu + \nu < N$, and $d_{\mu\nu} \in \widehat{A}$ for $\mu + \nu = N$, such that

$$a = \sum_{\mu+\nu < N} \psi(b_{\mu\nu}) f_0^\mu f_1^\nu + \sum_{\mu+\nu=N} d_{\mu\nu} f_0^\mu f_1^\nu. \quad (27)$$

Then, for all μ, ν with $\mu + \nu = N$ we obtain from Corollary 6.7 elements $b_{\mu\nu} \in B$ and $d_{\mu\nu}^0, d_{\mu\nu}^1 \in \widehat{A}$ such that

$$d_{\mu\nu} = \psi(b_{\mu\nu}) + d_{\mu\nu}^0 f_0 + d_{\mu\nu}^1 f_1.$$

Plugging into (27) yields

$$a = \sum_{\mu+\nu \leq N} \psi(b_{\mu\nu}) f_0^\mu f_1^\nu + \sum_{\mu+\nu=N} (d_{\mu\nu}^0 f_0 + d_{\mu\nu}^1 f_1) f_0^\mu f_1^\nu,$$

which is of the form (27) for $N+1$ and hence completes the induction. We have $\sum_{\mu, \nu \geq 0} \psi(b_{\mu\nu}) f_0^\mu f_1^\nu = a$, since this equality holds modulo \widehat{I}^N for all $N \geq 1$.

Now assume that $a \in A$. Then we claim

$$\deg_{\lambda\eta}(d_{\mu\nu}) \leq \gamma^{\mu+\nu} \deg a \quad \text{for all } \mu + \nu = N, \quad (28)$$

which for $N = 1$ follows directly from (25). Assuming (28) for some $N \geq 1$, (25) implies

$$\deg_{\lambda\eta} d_{\mu\nu}^0 \leq \max\{\deg_{\lambda+1, \eta} d_{\mu\nu}, \gamma \deg_{00} d_{\mu\nu}\} \leq \gamma^{\mu+\nu+1} \deg a,$$

and $d_{\mu\nu}^0$ contributes to $d_{\mu+1, \nu}$. A similar estimate holds for $d_{\mu\nu}^1$, which completes the proof of (28).

Finally, (28) and (24) yield

$$\deg(b_{\mu\nu}) \leq \deg_{00} d_{\mu\nu} \leq \gamma^{\mu+\nu} \deg a \quad \text{for all } \mu, \nu \in \mathbb{N}. \quad \square$$

Proof of Theorem 6.3. We prove the Theorem from scratch by constructing the residue map and checking that it is an isomorphism.

W.l.o.g. we can assume $p \geq 2$. As stated in the Theorem, we identify

$$H_{\text{dR}}^{p+1}(W) \simeq H^p \left(\frac{\Omega_{A_{f_0 f_1}}^\bullet}{\Omega_{A_{f_0}}^\bullet + \Omega_{A_{f_1}}^\bullet} \right).$$

Consider the map

$$\lambda: H_{\text{dR}}^{p-2}(Z) \rightarrow H^p \left(\frac{\Omega_{A_{f_0 f_1}}^\bullet}{\Omega_{A_{f_0}}^\bullet + \Omega_{A_{f_1}}^\bullet} \right), \quad [\bar{\omega}] \mapsto \left[\frac{df_0}{f_0} \wedge \frac{df_1}{f_1} \wedge \omega \right],$$

where $\omega \in \Omega_A^{p-2}$, $\bar{\omega}$ denotes its image in Ω_B^{p-2} , and $[\bar{\omega}]$ the cohomology class in $H_{\text{dR}}^{p-2}(Z)$ it represents. A similar notation is used on the right hand side.

We first show that the map λ is well-defined. If $\omega \in \Omega_A^{p-2}$ represents the zero cohomology class in $H_{\text{dR}}^{p-2}(Z)$, then $\bar{\omega} = d\bar{\eta}$ for some $\eta \in \Omega_A^{p-3}$. This means that $\omega - d\eta$ is contained in the differential graded ideal generated by I and dI . Using the formula $df_i \wedge \alpha = d(f_i \alpha) - f_i d\alpha$, we can assume that there exist $\alpha, \beta \in \Omega_A^{p-2}$ with $\omega - d\eta = f_0 \alpha + f_1 \beta$. Then

$$\begin{aligned} \frac{df_0}{f_0} \wedge \frac{df_1}{f_1} \wedge \omega &= df_0 \wedge \frac{df_1}{f_1} \wedge \alpha + \frac{df_0}{f_0} \wedge df_1 \wedge \beta + \frac{df_0}{f_0} \wedge \frac{df_1}{f_1} \wedge d\eta \\ &\equiv d \left(\frac{df_0}{f_0} \wedge \frac{df_1}{f_1} \wedge \eta \right) \bmod \left(\Omega_{A_{f_0}}^\bullet + \Omega_{A_{f_1}}^\bullet \right), \end{aligned}$$

so it maps to zero in the cohomology on the right hand side. Furthermore, since df_i/f_i is exact, one easily checks that λ sends closed (exact) forms to closed (exact) ones.

The residue map will be the inverse of λ . To construct it, note that the isomorphism of Lemma 6.8 induces a homomorphism $\Omega_{A_{f_0 f_1}}^\bullet \hookrightarrow \hat{\Omega}_{\hat{A}_{f_0 f_1}}^\bullet \xrightarrow{\simeq} \Omega_{B[[T_0, T_1]]_{T_0 T_1}}^\bullet$ and as a result an isomorphism

$$\vartheta: \frac{\Omega_{A_{f_0 f_1}}^\bullet}{\Omega_{A_{f_0}}^\bullet + \Omega_{A_{f_1}}^\bullet} \simeq \frac{\hat{\Omega}_{\hat{A}_{f_0 f_1}}^\bullet}{\hat{\Omega}_{\hat{A}_{f_0}}^\bullet + \hat{\Omega}_{\hat{A}_{f_1}}^\bullet} \xrightarrow{\simeq} \frac{\hat{\Omega}_{B[[T_0, T_1]]_{T_0 T_1}}^\bullet}{\hat{\Omega}_{B[[T_0, T_1]]_{T_0}}^\bullet + \hat{\Omega}_{B[[T_0, T_1]]_{T_1}}^\bullet}.$$

Define

$$\text{Res}: H^p \left(\frac{\Omega_{A_{f_0 f_1}}^\bullet}{\Omega_{A_{f_0}}^\bullet + \Omega_{A_{f_1}}^\bullet} \right) \longrightarrow H_{\text{dR}}^{p-2}(Z)$$

as follows. For a form $\omega \in \Omega_{A_{f_0 f_1}}^p$ write

$$\vartheta(\bar{\omega}) = \sum_{\mu, \nu \geq 1} (\alpha_{\mu\nu} + \beta_{\mu\nu} \wedge dT_0 + \gamma_{\mu\nu} \wedge dT_1 + \delta_{\mu\nu} \wedge dT_0 \wedge dT_1) T_0^{-\mu} T_1^{-\nu} \quad (29)$$

with $\alpha_{\mu\nu}, \beta_{\mu\nu}, \gamma_{\mu\nu}, \delta_{\mu\nu} \in \Omega_B^\bullet$, only finitely many non-zero, and map $\text{Res}([\bar{\omega}]) := [\delta_{1,1}]$. For the proof that this maps closed forms to closed forms, see below.

To prove that this defines the inverse of λ , it suffices to show $\text{Res} \circ \lambda = \text{id}$ and $\lambda \circ \text{Res} = \text{id}$ (in particular, this implies well-definedness). First, for $[\bar{\omega}] \in H_{\text{dR}}^{p-2}(Z)$ we have

$$\text{Res} \circ \lambda([\bar{\omega}]) = \text{Res} \left(\left[\frac{df_0}{f_0} \wedge \frac{df_1}{f_1} \wedge \omega \right] \right) = [\bar{\omega}],$$

since $\vartheta\left(\overline{\frac{df_0}{f_0} \wedge \frac{df_1}{f_1} \wedge \omega}\right) = \overline{\frac{dT_0}{T_0} \wedge \frac{dT_1}{T_1} \wedge \bar{\omega}}$.

On the other hand, let $\omega \in \Omega_{A_{f_0 f_1}}^p$ be a form with $d\omega \in \Omega_{A_{f_0}}^p + \Omega_{A_{f_1}}^p$. Writing $\vartheta(\overline{\omega})$ in the form (29) and differentiating yields

$$\begin{aligned}
d\vartheta(\overline{\omega}) &= \sum_{\mu, \nu \geq 1} \left((d\alpha_{\mu\nu} + d\beta_{\mu\nu} \wedge dT_0 + d\gamma_{\mu\nu} \wedge dT_1 + d\delta_{\mu\nu} \wedge dT_0 \wedge dT_1) T_0^{-\mu} T_1^{-\nu} \right. \\
&\quad - (-1)^p \mu (\alpha_{\mu\nu} + \gamma_{\mu\nu} \wedge dT_1) T_0^{-\mu-1} T_1^{-\nu} dT_0 \\
&\quad \left. - (-1)^p \nu (\alpha_{\mu\nu} + \beta_{\mu\nu} \wedge dT_0) T_0^{-\mu} T_1^{-\nu-1} dT_1 \right) \\
&= \sum_{\mu, \nu \geq 1} T_0^{-\mu} T_1^{-\nu} \left(d\alpha_{\mu\nu} + (d\beta_{\mu\nu} + (-1)^{p+1} (\mu-1) \alpha_{\mu-1, \nu}) \wedge dT_0 \right. \\
&\quad + (d\gamma_{\mu\nu} + (-1)^{p+1} (\nu-1) \alpha_{\mu, \nu-1}) \wedge dT_1 \\
&\quad \left. + (d\delta_{\mu\nu} + (-1)^p (\mu-1) \gamma_{\mu-1, \nu} + (-1)^{p+1} (\nu-1) \beta_{\mu, \nu-1}) \wedge dT_0 \wedge dT_1 \right).
\end{aligned}$$

Since in all terms of this expression both T_0 and T_1 have negative exponents and are contained in $\widehat{\Omega}_{B[[T_0, T_1]]_{T_0}}^p + \widehat{\Omega}_{B[[T_0, T_1]]_{T_1}}^p$, they must be zero in $\widehat{\Omega}_{B[[T_0, T_1]]_{T_0 T_1}}^p$. Among others, this implies the relations

$$d\beta_{\mu\nu} + (-1)^{p+1} (\mu-1) \alpha_{\mu-1, \nu} = 0, \quad (30)$$

$$d\delta_{\mu\nu} + (-1)^p (\mu-1) \gamma_{\mu-1, \nu} + (-1)^{p+1} (\nu-1) \beta_{\mu, \nu-1} = 0 \quad (31)$$

for all $\mu, \nu \geq 1$. In particular, (31) shows that $\delta_{1,1}$ is closed. We have to show that all other terms of ω can be integrated. To do so, define

$$\eta := \sum_{\mu \geq 2} \sum_{\nu \geq 1} (-1)^p \frac{T_0^{1-\mu}}{1-\mu} T_1^{-\nu} (-\beta_{\mu\nu} + \delta_{\mu\nu} \wedge dT_1) + \sum_{\nu \geq 2} (-1)^{p+1} T_0^{-1} \frac{T_1^{1-\nu}}{1-\nu} \delta_{1, \nu} \wedge dT_0.$$

We check

$$\begin{aligned}
d\eta &= \sum_{\mu \geq 2} \sum_{\nu \geq 1} (-1)^p \left((T_0^{-\mu} T_1^{-\nu} dT_0 - \frac{T_0^{1-\mu}}{1-\mu} \nu T_1^{-\nu-1} dT_1) \wedge (-\beta_{\mu\nu} + \delta_{\mu\nu} \wedge dT_1) \right. \\
&\quad + \frac{T_0^{1-\mu}}{1-\mu} T_1^{-\nu} (-d\beta_{\mu\nu} + d\delta_{\mu\nu} \wedge dT_1) \Big) \\
&\quad + \sum_{\nu \geq 2} (-1)^{p+1} \left((-T_0^{-2} \frac{T_1^{1-\nu}}{1-\nu} dT_0 + T_0^{-1} T_1^{-\nu} dT_1) \wedge \delta_{1, \nu} \wedge dT_0 \right. \\
&\quad \left. + T_0^{-1} \frac{T_1^{1-\nu}}{1-\nu} d\delta_{1, \nu} \wedge dT_0 \right) \\
&= \sum_{\mu \geq 2} \sum_{\nu \geq 1} \left((-1)^{p+1} \frac{T_0^{1-\mu}}{1-\mu} T_1^{-\nu} d\beta_{\mu\nu} + T_0^{-\mu} T_1^{-\nu} \beta_{\mu\nu} \wedge dT_0 \right. \\
&\quad + \frac{T_0^{1-\mu}}{1-\mu} (-\nu T_1^{-\nu-1} \beta_{\mu\nu} + (-1)^p T_1^{-\nu} d\delta_{\mu\nu}) \wedge dT_1 + T_0^{-\mu} T_1^{-\nu} \delta_{\mu\nu} dT_0 \wedge dT_1 \Big) \\
&\quad + \sum_{\nu \geq 2} T_0^{-1} \left(T_1^{-\nu} \delta_{1, \nu} \wedge dT_0 \wedge dT_1 + (-1)^{p+1} \frac{T_1^{1-\nu}}{1-\nu} d\delta_{1, \nu} \wedge dT_0 \right).
\end{aligned}$$

Using (30) and (31) for $\mu = 1$ we obtain

$$\begin{aligned}
d\eta &= \sum_{\mu \geq 2} \sum_{\nu \geq 1} \left(T_0^{1-\mu} T_1^{-\nu} \alpha_{\mu-1, \nu} + T_0^{-\mu} T_1^{-\nu} \beta_{\mu\nu} \wedge dT_0 \right. \\
&\quad \left. + \frac{T_0^{1-\mu}}{1-\mu} T_1^{-\nu} (-(\nu-1)\beta_{\mu, \nu-1} + (-1)^p d\delta_{\mu\nu}) \wedge dT_1 \right) \\
&\quad + \sum_{\substack{\mu, \nu \geq 1 \\ (\mu, \nu) \neq (1, 1)}} T_0^{-\mu} T_1^{-\nu} \delta_{\mu\nu} dT_0 \wedge dT_1 + \sum_{\nu \geq 2} T_0^{-1} T_1^{1-\nu} \beta_{1, \nu-1} \wedge dT_0 \\
&\stackrel{(31)}{=} \sum_{\mu \geq 1} \sum_{\nu=1}^s T_0^{-\mu} T_1^{-\nu} (\alpha_{\mu, \nu} + \beta_{\mu\nu} \wedge dT_0) + \sum_{\mu \geq 2} \sum_{\nu \geq 1} T_0^{1-\mu} T_1^{-\nu} \gamma_{\mu-1, \nu} \wedge dT_1 \\
&\quad + \sum_{\substack{\mu, \nu \geq 1 \\ (\mu, \nu) \neq (1, 1)}} T_0^{-\mu} T_1^{-\nu} \delta_{\mu\nu} dT_0 \wedge dT_1 \\
&= \vartheta(\bar{\omega}) - T_0^{-1} T_1^{-1} \delta_{1,1} dT_0 \wedge dT_1,
\end{aligned}$$

thus $\lambda \circ \text{Res}([\bar{\omega}]) = [\bar{\omega}]$.

In order to prove (15), we have to bound the degree of $\delta_{1,1}$ in (29). Note that by linearity it suffices to consider terms of the form

$$\omega = \frac{a}{(f_0 f_1)^s} dX_{i_1} \wedge \cdots \wedge dX_{i_p}, \quad a \in A, \quad 0 \leq i_1 < \cdots < i_p \leq n, \quad s \geq 1.$$

By Lemma 6.8 we have

$$\begin{aligned}
b &:= \hat{\psi}^{-1}(a) = \sum_{\mu, \nu \geq 0} b_{\mu\nu} T_0^\mu T_1^\nu, \\
\Xi_i &:= \hat{\psi}^{-1}(X_i) = \sum_{\mu, \nu \geq 0} b_{\mu\nu}^{(i)} T_0^\mu T_1^\nu \in B[[T_0, T_1]],
\end{aligned}$$

where

$$\deg b_{\mu\nu} \leq \gamma^{\mu+\nu} \deg a, \quad \deg b_{\mu\nu}^{(i)} \leq \gamma^{\mu+\nu}. \quad (32)$$

It follows

$$\vartheta(\bar{\omega}) = \frac{b}{(T_0 T_1)^s} d\Xi_{i_1} \wedge \cdots \wedge d\Xi_{i_p}.$$

Moreover, we have

$$\begin{aligned}
d\Xi_i &= \sum_{\mu, \nu \geq 0} (db_{\mu\nu}^{(i)} T_0^\mu T_1^\nu + b_{\mu\nu}^{(i)} \mu T_0^{\mu-1} T_1^\nu dT_0 + b_{\mu\nu}^{(i)} \nu T_0^\mu T_1^{\nu-1} dT_1) \\
&= \sum_{\mu, \nu \geq 0} (db_{\mu\nu}^{(i)} + (\mu+1)b_{\mu+1, \nu}^{(i)} dT_0 + (\nu+1)b_{\mu, \nu+1}^{(i)} dT_1) T_0^\mu T_1^\nu.
\end{aligned}$$

The terms of $\vartheta(\bar{\omega})$ involving $dT_0 \wedge dT_1$ are of the form

$$\begin{aligned}
&\pm (\mu_1 + 1)(\nu_2 + 1) b_{\mu_1+1, \nu_1}^{(i)} b_{\mu_2, \nu_2+1}^{(j)} T_0^{\mu_1+\mu_2+\cdots+\mu_p-s} T_1^{\nu_1+\nu_2+\cdots+\nu_p-s} \\
&\quad \cdot dT_0 \wedge dT_1 \wedge d_B b_{\mu_3, \nu_3}^{(j_1)} \wedge \cdots \wedge d_B b_{\mu_p, \nu_p}^{(j_{p-2})}
\end{aligned}$$

with some $0 \leq i, j, j_1, \dots, j_{p-2} \leq n$ and $\mu, \nu, \mu_1, \nu_1, \dots, \mu_p, \nu_p \geq 0$. To get the coefficient $\delta_{1,1}$ of $dT_0/T_0 \wedge dT_1/T_1$, we have to consider the case $\mu + \mu_1 + \dots + \mu_p = s - 1$ and $\nu + \nu_1 + \dots + \nu_p = s - 1$. Using that d_B is of degree 0 together with the estimate (32), it follows that $\delta_{1,1}$ is of degree

$$\begin{aligned} &\leq \deg b_{\mu\nu} + \deg b_{\mu_1+1, \nu_1}^{(i)} + \deg b_{\mu_2, \nu_2+1}^{(j)} + \deg b_{\mu_3, \nu_3}^{(j_1)} + \dots + \deg b_{\mu_p, \nu_p}^{(j_{p-2})} \\ &\leq \gamma^{\mu+\nu} \deg a + \gamma^{\mu_1+\nu_1+1} + \gamma^{\mu_2+\nu_2+1} + \gamma^{\mu_3+\nu_3} + \dots + \gamma^{\mu_p+\nu_p} \\ &\leq \gamma^{2s-1} (\deg a + p). \end{aligned} \quad \square$$

7 Proof of the Main Theorem

The effective Gysin sequence yields degree and order bounds for the de Rham cohomology of a smooth hypersurface.

Theorem 7.1. *Let $f, g \in R = k[X_1, \dots, X_n]$ with $d := \deg f$ and $d' := \deg g$, such that f is irreducible and $\frac{\partial f}{\partial X_n} | g$, and consider the smooth hypersurface $V := \mathcal{Z}(f) \setminus \mathcal{Z}(g) \subseteq \mathbb{A}^n$. Then we have*

$$(p+2)(d+d'+2)(2d'-d+3)^{2p+3} \cdot (1,1) \in B(H_{\text{dR}}^p(V)).$$

for all $p \in \mathbb{N}$.

Proof. Putting $f_0 := gX_0 - 1$, $f_1 := f$, we have the isomorphism

$$V \xrightarrow{\sim} Z := \mathcal{Z}(f_0, f_1) \subseteq \mathbb{A}^{n+1}, \quad x \mapsto (1/g(x), x),$$

and the pull-back of differential forms shows

$$\deg(H_{\text{dR}}^p(Z)) \cdot (1,1) \in B(H_{\text{dR}}^p(V)).$$

Thus, we have reduced to the setting of Theorem 6.3. Note that $d_0 := \deg f_0 = d' + 1$ and $d_1 := \deg f_1 = d$. By Corollary 5.6 we have

$$p \cdot (1,1) \in B \left(H^p \left(\frac{\Omega_{A_{f_0 f_1}}^\bullet}{\Omega_{A_{f_0}}^\bullet + \Omega_{A_{f_1}}^\bullet} \right) \right).$$

Theorem 6.3 implies

$$\deg(H_{\text{dR}}^{p-2}(Z)) \leq (2d_0 - d_1 + 1)^{2p-1} p(d_0 + d_1 + 1) = p(d+d'+2)(2d'-d+3)^{2p-1},$$

which implies the claim. \square

Proof of Theorem 1.1. Let $X \subseteq \mathbb{A}^n$ be a smooth closed variety. The zeroth cohomology is treated in §3, and the case $D = 1$ is trivial, so we assume $n > m \geq 1$ and $D \geq 2$.

First assume that X is irreducible. Then, by Corollary 4.5 and Lemma 4.2 we can write $X = \bigcup_{i=0}^m U_i$, where $U_i = X \setminus \mathcal{Z}(g_i)$ is isomorphic to $V_i = Y_i \setminus \mathcal{Z}(g_i^*) \subseteq$

\mathbb{A}^{m+1} . Furthermore, in suitable coordinates X_1, \dots, X_n of \mathbb{A}^n , the isomorphism is given by the projection $\pi: U_i \rightarrow V_i$ onto the first $m+1$ coordinates, and we have $Y_i = \mathcal{Z}(f_i)$, where $f_i \in k[X_1, \dots, X_{m+1}]$ is irreducible and monic in X_{m+1} , $g_i^* = g_i = \frac{\partial f_i}{\partial X_{m+1}}$, and $\deg f_i \leq D$. Now fix $1 \leq \ell \leq m$. For $p, q \in \mathbb{N}$ with $p+q = \ell$ consider a multi-index $0 \leq i_0 < \dots < i_q \leq m$. Set $i := i_0$, $U' := U_{i_0 \dots i_q}$, and $V' := \pi(U')$. Note that $U' = U_i \setminus \mathcal{Z}(h)$ with $h := g_{i_1} \dots g_{i_q}$. From Corollary 4.4 we obtain $H \in k[Y_i]$ with $\deg H \leq D \deg h$ such that $k[V'] = k[Y_i]_{g_i H}$. We can assume $d := \deg f_i \geq 2$, and with $g := g_i H$ we have $d-1 \leq d' := \deg g \leq d-1 + \deg H \leq (D-1)(qD+1) \leq \ell D^2 + D - 2$. Theorem 7.1 implies

$$(p+2)(d+d'+2)(2d'-d+3)^{2p+3} \cdot (1,1) \in B(H_{\text{dR}}^p(V)).$$

Moreover,

$$\begin{aligned} & (p+2)(d+d'+2)(2d'-d+3)^{2p+3} \\ & \leq (p+2)(D+(D-1)(qD+1)+2)(2(D-1)(qD+1)+1)^{2p+3} \\ & \leq (p+2)(\ell D^2+2D)(2D(q(D-1)+1))^{2p+3} \\ & \leq 6\ell^2 D^2 (2\ell D^2)^{2\ell+3} = 3 \cdot 2^{2\ell+4} \ell^{2\ell+5} D^{4\ell+8}. \end{aligned}$$

Using Corollary 4.4 we conclude

$$(s, d) := 3 \cdot 2^{2\ell+4} \ell^{2\ell+5} D^{4\ell+8} \cdot (\ell D, 1) \in B(H_{\text{dR}}^p(U_{i_0 \dots i_q})), \quad (33)$$

Since this holds for all $p+q = \ell$ and all $i_0 < \dots < i_q$, Theorem 5.3 shows that

$$\begin{aligned} \deg H_{\text{dR}}^\ell(X) & \leq d + 2D(\ell+1)(s+\ell)^m(D-1)^m \\ & \leq 2D(\ell+1)(\ell Dd + \ell)^m D^m \\ & \leq 4\ell^{m+1}(Dd+1)^m D^{m+1} \\ & \stackrel{(33)}{\leq} 4\ell^{m+1}(2^{2\ell+6} \ell^{2\ell+5} D^{4\ell+9})^m D^{m+1} \\ & = 2^{2\ell m+6m+2} \ell^{2\ell m+6m+1} D^{4\ell m+10m+1}. \end{aligned}$$

Finally, for possibly reducible X , Corollary 4.1 implies the claim. \square

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